Chapter 9

Analytic Systems

§ 9.0 *Introduction*

After a good understanding of thermodynamics and thermochemistry was achieved by the middle of the 19th century, and before the advent of fast computing power in the middle of the 20th century, a great deal of ingenuity was expended on developing closed mathematical functions to describe the progress of the projectile (principally artillery shells) down the barrel of a gun, starting with the seminal work of the French ballistician Sarreau in 1876/7 [1]. From the beginning however, there were two objectives for the attempted solutions, which were to some extent diametrically opposed. In one camp were the "mathematical ballisticians", who sought general "exact" solutions to the basic internal ballistic equations in order to achieve a proper understanding of the inter-relationship of these equations and their various parameters. In the other camp were the "practical ballisticians", who were prepared to admit approximations and simplifications to achieve working solutions that were accurate enough to be useful, but requiring the least amount of computational effort.

In the first camp was (for example) the work of Kapur [2] and Struble [3] who built on the early work of Clemmow [4] to develop generalised solutions for the ballistic equations as far as possible. In the second camp, the systems best known (and still used to check numerical models today) are those of Coppock [5], Goldie [6] and Corner [7], principally because they were described in the classic text book on internal ballistics by Corner. Semi-graphical systems due to Hunt-Hinds is described in "Interior Ballistics" [8] produced in 1951, and that of Taylor is described in "Interior Ballistics of Guns" [9], a handbook produced by the US Army in 1965. Excellent reviews of the various campaigns to develop analytical solutions, together with an extensive bibliography, is given by Bhaskara Rao and Sharma [10]. They also describe and reference much work that was done in the Soviet Union in a parallel but separate effort to the work in the West.

Analytic systems which attempt to be exact very quickly assume a complexity which was acceptable in terms of computational time only when there was no other recourse to attaining a solution. But with the advances in fast, cheap computing power during the 1950s and '60s, it became much easier to solve the ballistic equations numerically, and without having to sacrifice exactness by approximations or simplifications. Today, there is no longer any practical interest in these complex analytic systems.

However, it remains true that analytic models can give an insight into the interrelations between the various parameters, and how they scale against each other, in a way which is not apparent in numerical systems. They can also give a benchmark for the accuracy of numerical systems which are set up for the same initial conditions and for which the analytic system gives the nominally correct solutions. To this end, it is worth looking in more detail at the elegant system proposed by Mayer and Hart [11], which curiously was not reviewed by Bhaskara Rao and Sharma, but which was used for many years in various guises at the Ballistics Research Laboratory in the USA, where it was considered a very reliable predictor of muzzle velocity. Baer [12] reported that using data from 96 guns ranging in size from .30 calibre to 280mm cannon, the Mayer-Hart method predicted the muzzle velocity with an average error of only 1.77%.

Indeed Baer, who developed the first numerical internal ballistics programs to be run on a digital computer, considered that the Mayer-Hart system was still the best general predictor of muzzle velocity. He reasoned that though more complex numerical systems can account specifically for shot-start pressure, barrel friction, heat losses and so on, these factors were actually not that well understood *a-priori* before testing a particular gun, so that these more sophisticated systems did not actually do any better than a simple system where these factors were accounted for with a broader brush.

§ 9.1 *The Mayer-Hart system*

The Mayer-Hart system is described as a set of 'simplified' equations of internal ballistics in that a number of approximations are made so that the ballistic equations can be solved in a closed manner. The derivation given here differs slightly from that in the original paper, in that it gives a somewhat less opaque path through the mathematical thickets and brambles *en-route* to the solutions.

§ 9.2 *The approximation used in the Mayer-Hart system*

1) **The shot start pressure is zero.** That is, there is no separate variable for shot start pressure and it is assumed that the projectile starts to move immediately the powder starts to burn. This assumption is not really significant (in that it does not affect the maximum chamber pressure or muzzle velocity significantly) for large guns and cannons, or for the larger calibres and magnums in small-arms rifles, where the rise time to maximum pressure is much longer than the time taken for the projectile to engrave the rifling. It is of considerable significance for pistols, revolvers and the smaller rifle calibres which use very fast powders and where the pressure rise times are comparable to the time it takes for the projectile to engrave the rifling.

2) **The covolume of the propellant gasses is equal to the reciprocal of the original charge density.** In fact, the covolume of the propellant gasses is about 50% greater than the reciprocal of the original charge volume. This approximation is of no consequence when it matters, at the start of the projectile's journey down the barrel, when the pressure is still rising quickly to maximum pressure and relatively little of the powder has burnt. It is also not significant when the projectile is well advanced down the barrel, at which time the system has effectively lost memory of the starting conditions anyway. It is of consequence around the time of maximum pressure, when the expansion ratio has not increased much and a significant amount of the propellant has already burnt.

3) **The burning rate of the propellant is linearly proportional to the pressure.** This has been found to be a reasonable approximation in practice where linear burn rates are determined for the pressures at which firearms generally operate, and is in any case an assumption used even in sophisticated numerical systems.

4) **The area of the burning surface remains constant.** This assumption is approximately true for "neutral" powder kernel forms, such as flakes and cylindrical powder kernels with one perforation, which are the most common powder kernel shapes in small-arms powders. Spherical ball powders have a naturally digressive shape, but this is mitigated by deterrent coatings which make such powders burn in more of a neutral or even slightly progressive manner. Progressive shapes such as multi-perforated kernels have been found to splinter relatively early in their burning cycle, so that they tend to burn in a more neutral manner than their shape would suggest. Generally then, this is a reasonable approximation,

5) **That no energy is lost through friction as the projectile travels up the barrel.** There is no separate variable for friction. This can be accounted for by increasing the effective mass of the projectile.

6) **That no energy is lost through heat to the walls of the chamber or barrel.** There is no separate variable for heat loss. This can be accounted for by increasing the value of gamma, the ratio of specific heats, which mimics the effect of heat loss by reducing the thermodynamic efficiency of the system.

7) **The projectile base pressure is the same as the chamber pressure** The gradient in gas pressure in the gas column behind the projectile (see Chapter 8) means that the pressure accelerating the projectile down the barrel is not the same as the pressure at which the propellant is said to be burning. (See Chapter 4). This difference in pressure is commonly accounted for by introducing the Lagrange pressure relationship or increasing the effective mass of the projectile.

§ 9.3 *The principal internal ballistics equations*

In chapter 4 it was described how the propellant burns away from the outside layer of the powder kernels. If C is the original charge weight of propellant, and Z is the fraction of propellant that has burnt away, then the mass of propellant that has been converted to propellant gasses is CZ. In section § 4.6 it was shown that for propellant powders with a neutral kernel shape, the rate at which the propellant mass burns away could be written as,

$$\frac{dZ}{dt} = \frac{2\beta P^{\alpha}}{w}$$

Where β is the burning rate coefficient for the propellant, w is the 'web thickness' and P is the pressure. It is assumed in this case that burning rate is proportional to the pressure, so α here is one. Then,

$$\frac{dZ}{dt} = \frac{2\beta P}{w} \qquad 9.1$$

This is the first of the internal ballistics equations.

Let V_C be the initial volume in the case or chamber behind the loaded projectile. Then,

$$V_0 = V_C - \frac{C}{\rho} \qquad 9.2$$

Where V_0 is the free volume that is left in the chamber when a charge C of propellant with density ρ has been loaded.

Once the propellant starts burning and the projectile begins to move up the barrel, the volume behind the projectile will be V where,

$$V = V_0 + Ax \qquad 9.3$$

The area of the bore is A and the distance the projectile has travelled up the barrel is x.

Résal's equation (see § 3.5) can be written as,

$$PV = CZF - (\gamma - 1)\left(\frac{m_{\text{eff}}\, v^2}{2g}\right) \qquad 9.4$$

The effective projectile mass here is m_{eff} (see § 9.10.2), its velocity up the barrel is v and the Force of the propellant is F. This is the second ballistic equation.

If the projectile did not move, then the chamber would effectively form a closed bomb. The maximum pressure would be P_C where $P_C V_0 = C F$. Let P_C be a ballistic coefficient.

The equation of motion will be the third ballistic equation where,

$$PA = \frac{m_{\text{eff}}}{g} \frac{dv}{dt} \qquad 9.5$$

§ 9.4 *The solutions to the internal ballistics equations*

The aim now is to find the pressure as a function of distance the projectile has moved up the barrel. This distance is made dimensionless by transforming it into an expansion ratio $\left(V/V_0\right)$. To start then, Résal's equation is divided through by V_0 so that,

$$P\left(\frac{V}{V_0}\right) = \frac{CZF}{V_0} - \frac{(\gamma - 1)}{V_0}\left(\frac{m_{\text{eff}} v^2}{2g}\right) = P_C Z\left[1 - \frac{(\gamma-1)}{CZF}\frac{m_{\text{eff}} v^2}{2g}\right] \qquad 9.6$$

Now, from Eqn. 9.5,

$$v = \frac{Ag}{m_{\text{eff}}}\int P\,dt \qquad 9.7$$

and from Eqn. 9.1,

$$Z = \frac{2\beta}{w}\int P\,dt \qquad 9.8$$

Eqn. 9.7 and Eqn. 9.8 can be combined so that,

$$v = \frac{AgZw}{2\beta m_{\text{eff}}} \qquad 9.9$$

Note that the projectile velocity is here a linear function of Z. This means that while the powder is burning, the projectile velocity is proportional to the energy in the propellant gasses.

Substituting for v in Eqn. 9.6,

$$P\left(\frac{V}{V_0}\right) = P_C Z\left[1 - \frac{(\gamma-1)}{2}\frac{A^2 g Z w^2}{4CF\beta^2 m_{\text{eff}}}\right] \qquad 9.10$$

If Eqn. 9.10 is rewritten as,

$$\frac{P}{P_C}\left(\frac{V}{V_0}\right) = Z\left[1 - \frac{(\gamma-1)}{2}\frac{A^2 g Z w^2}{4\,C\,F\,\beta^2 m_{\text{eff}}}\right] \qquad 9.11$$

then it is evident that Eqn. 9.11 is dimensionless. Gathering the various constants together then, a second ballistic coefficient P_Q can be defined such that,

$$\frac{A^2 g\, w^2}{4\,C\,F\,\beta^2 m_{\text{eff}}} = \left(\frac{P_C}{P_Q}\right) \quad \text{where,}\ P_Q = \left(\frac{2\,C\,F\,\beta}{A w}\right)^2 \frac{m_{\text{eff}}}{g\,V_0} \qquad 9.12$$

The ballistic coefficient P_Q has the dimensions of pressure.

Eqn. 9.10 can now be rewritten as,

$$P\left(\frac{V}{V_0}\right) = P_C Z\left[1 - \frac{(\gamma-1)}{2}\left(\frac{P_C}{P_Q}\right)Z\right] \qquad 9.13$$

Now, from Eqn. 9.5, the pressure P can be expressed as,

$$P = \frac{m_{\text{eff}}}{g\,A}\frac{dv}{dt} \qquad 9.14$$

Then, with some manipulation using the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = A v\frac{dv}{d(Ax)} = A v\frac{dv}{dV} = A v\frac{dv}{dZ}\frac{dZ}{dV} \qquad 9.15$$

From Eqn. 9.9,

$$\frac{dv}{dZ} = \frac{A g w}{2\beta\, m_{\text{eff}}} \qquad 9.16$$

Substituting Eqn. 9.15 and Eqn. 9.16 into Eqn. 9.14,

$$P = \left(\frac{A w}{2\beta}\right)^2 \frac{Z g}{m_{\text{eff}}}\frac{dZ}{dV} \rightarrow P_C\left(\frac{P_C}{P_Q}\right) Z V_0 \frac{dZ}{dV} \qquad 9.17$$

Substituting for P in Eqn. 9.13,

$$P_C\left(\frac{P_C}{P_Q}\right)ZV_0\frac{dZ}{dV}\left(\frac{V}{V_0}\right) = P_C Z\left[1 - \frac{(\gamma-1)}{2}\left(\frac{P_C}{P_Q}\right)Z\right] \qquad 9.18$$

Simplifying,

$$\frac{dZ}{dV}V = \left(\frac{P_Q}{P_C}\right) - \frac{(\gamma-1)}{2}Z \qquad 9.19$$

Inverting and rearranging,

$$\frac{dV}{V} = \left[\frac{P_Q}{P_C} - \frac{(\gamma-1)}{2}Z\right]^{-1} dZ \qquad 9.20$$

Integrating,

$$\int_{V_0}^{V}\left[\frac{dV'}{V'}\right] = \int_{0}^{Z}\left[\frac{P_Q}{P_C} - \frac{(\gamma-1)}{2}Z'\right]^{-1} dZ' \qquad 9.21$$

Then,

$$\ln\left(\frac{V}{V_0}\right) = -\frac{2}{(\gamma-1)}\ln\left[1 - \frac{(\gamma-1)}{2}\left(\frac{P_C}{P_Q}\right)Z\right] \qquad 9.22$$

Taking anti-logs,

$$\left(\frac{V}{V_0}\right) = \left[1 - \frac{(\gamma-1)}{2}\left(\frac{P_C}{P_Q}\right)Z\right]^{-\frac{2}{(\gamma-1)}} \qquad 9.23$$

The expansion ratio $\left(V/V_0\right)$ is now in terms of Z, the fraction of propellant which has burnt. Rearranging so that Z is in terms of the expansion ratio,

$$Z = \frac{2}{(\gamma-1)}\left(\frac{P_Q}{P_C}\right)\left[1 - \left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right] \qquad 9.24$$

§ 9.4.1 *Pressure as a function of expansion ratio*

Substituting for Z in Eqn. 9.13, an expression for pressure in terms of the expansion ratio (and so distance the projectile has travelled up the barrel) is achieved.

$$P = \frac{2P_Q}{(\gamma-1)}\left(\frac{V}{V_0}\right)^{-\gamma}\left[\left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}} - 1\right] \qquad 9.25$$

§ 9.4.2 *Kinetic energy as a function of expansion ratio*

The kinetic energy of the projectile and propellant gasses can be expressed from Eqn. 9.9 and substituting for Z from Eqn. 9.24.

$$\frac{m_{\text{eff}}\, v^2}{2} = \frac{Z^2 P_C V_0\, g}{2}\left(\frac{P_C}{P_Q}\right) \rightarrow \frac{2\, V_0\, P_Q}{(\gamma-1)^2}\left[1 - \left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right]^2 \qquad 9.26$$

§ 9.4.3 *Projectile velocity as a function of expansion ratio*

The projectile velocity as a function of expansion ratio is then,

$$v = \left(\frac{4gV_0P_Q}{m_{\text{eff}}}\right)^{\frac{1}{2}} \frac{1}{(\gamma - 1)}\left[1 - \left(\frac{V}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right] \qquad 9.27$$

§ 9.4.4 *Projectile position at maximum pressure*

Maximum pressure will occur when $\dfrac{dP}{d\left(V/V_0\right)} = 0$, So then,

$$\left(\frac{V}{V_0}\right)_{\text{P-Max}} = \left[\frac{2\gamma}{(\gamma + 1)}\right]^{\frac{2}{(\gamma-1)}} \qquad 9.28$$

This is an interesting result. Eqn. 9.28 shows that expansion ratio at maximum pressure does not depend on the projectile weight or the burning rate, but only on gamma, the ratio of specific heats. Sophisticated numerical systems also show a surprising invariance with projectile weight or the burning rate for the projectile travel distance to maximum pressure. Given that for most propellants $\gamma \approx 1.24$, within a few percent, the consequence is that for all cartridges across all calibres, and for all guns, the expansion ratio at which maximum pressure occurs is about 2.3.

§ 9.4.5 *Amount of charge burnt at maximum pressure*

Substituting Eqn. 9.28 into Eqn. 9.24 then,

$$Z_{\text{P-Max}} = \frac{P_Q}{\gamma P_C} \qquad 9.29$$

§ 9.4.6 *Value of maximum pressure*

At the position of maximum pressure from Eqn. 9.28, the value of the maximum pressure from Eqn. 9.25 will be,

$$P_{\text{Max}} = P_Q \left[(\gamma - 1)^{(\gamma+1)} \gamma^{-2\gamma} 2^{-(\gamma+1)} \right]^{\frac{1}{(\gamma-1)}} \qquad 9.30$$

This may be approximated to good accuracy (about one part in a thousand) by,

$$P_{\text{Max}} = \left(\frac{P_Q}{\mathrm{e}} \right) \left[1 + \frac{3}{4}(\gamma - 1) \right]^{-1} \qquad 9.31$$

Where e = 2.718 and is the base of the natural logarithm system.

§ 9.5 *Scaling factors that affect maximum pressure*

From Eqn. 9.30, maximum pressure is a linear function of the ballistic coefficient P_Q which can be written as,

$$P_Q = \left(\frac{2\,\mathrm{C} F \beta}{\mathrm{A} w} \right)^2 \frac{m}{g V_C} \left[1 - \frac{\mathrm{C}}{V_C \rho} \right]^{-1} \qquad 9.32$$

It can be seen that the maximum pressure will go as the burning rate β^2, on the web thickness w^{-2}, on the Force as F^2, and on the inverse of the calibre to the forth power. It is also proportional to the projectile weight and depends on the charge weight as, $\mathrm{C}^2/\left(1 - \mathrm{C}/\rho V_C\right)$. The dependence on the chamber volume is as $(V_C - \mathrm{C}/\rho)^{-1}$, which shows how the maximum chamber pressure will rise rapidly with, and is sensitive to, the load when the load is near to filling the case or chamber.

§ 9.6 *Vivacity*

The concept of vivacity was introduced in section §4.8. The vivacity Λ is the fractional rate at which the powder volume burns away per unit of pressure, so that $d\mathrm{Z}/dt = \Lambda P$ From Eqn. 9.1, $\Lambda = 2\beta/w$ and the expression for P_Q in Eqn. 9.12 can be replaced by,

$$P_Q = \left(\frac{CF\Lambda}{\text{A}} \right)^2 \frac{m_{\text{eff}}}{g V_0} \qquad 9.33$$

Note that the web thickness w has dropped out of the equation. If vivacity is used instead of burning rate, then there is no need for any knowledge of the powder kernel geometry. Note too that the maximum pressure will scale as the square of the vivacity.

The average vivacities for a number of commercially available propellant powders are given in Appendix 2.

§ 9.7 *When the propellant is 'all-burnt'*

The propellant is all-burnt when Z = 1 and from Eqn. 9.23, this happens when the expansion ratio has reached a value of,

$$\left(\frac{V}{V_0} \right) = \left[1 - \frac{(\gamma - 1)}{2} \left(\frac{P_C}{P_Q} \right) \right]^{-\frac{2}{(\gamma-1)}} \qquad 9.34$$

This is an under-estimation of the position of all burnt in real guns, since propellants do not burn at a constant rate to the end of the burning cycle. Due to fracturing and splintering, the propellant kernels disintegrate into pieces which burn at a divergent rate, which peters away to zero in the final stages of burning.

The pressure at all-burnt is,

$$P_{\text{All-Burnt}} = P_C \left[1 - \frac{(\gamma - 1)}{2} \left(\frac{P_C}{P_Q} \right) \right]^{-\frac{(\gamma+1)}{(\gamma-1)}} \qquad 9.35$$

The velocity at all-burnt is,

$$v_{\text{All-Burnt}} = \left[\frac{CFg}{m_{\text{eff}}} \left(\frac{P_C}{P_Q} \right) \right]^{\frac{1}{2}} \qquad 9.36$$

§ 9.8 *Beyond 'all-burnt', muzzle velocity and muzzle pressure*

After all-burnt, the amount of charge burnt does not increase, so it must be true that Z equals unity thereafter. However, the value of Z in Eqn. 9.24 does not stop at Z = 1 as the expansion ration increases, so the equations above are not valid beyond the expansion ratio where all-burnt is reached. It is usual in all analytic systems to proceed from here on the basis that the volume behind the projectile is expanding so quickly that it can be treated as adiabatic, i.e. that is there are no heat losses to the walls.

Assuming that all burnt happens before the projectile has reached the muzzle, the pressure at the muzzle P_M can be related to the pressure at all-burnt using the equation of adiabatic expansion $P_0 V_0^{\gamma} = P_1 V_1^{\gamma}$ (see § 3.3).

$$P_M = P_C \left(\frac{V_0}{V_M}\right)^{\gamma} \left[1 - \frac{(\gamma-1)}{2}\left(\frac{P_C}{P_Q}\right)\right]^{-1} \qquad 9.37$$

The muzzle velocity can obtained using Resal's equation (Eqn. 9.4) so that,

$$v_M = \left(\frac{2CFg}{m_{eff}(\gamma-1)}\left\{1 - \left(\frac{V_0}{V_M}\right)^{(\gamma-1)}\left[1 - \frac{(\gamma-1)}{2}\left(\frac{P_C}{P_Q}\right)\right]^{-1}\right\}\right)^{\frac{1}{2}} \qquad 9.38$$

These equations are of course applicable for any position past all-burnt, as well as at the muzzle.

It can be seen here that for 'long' barrels, where $V \gg V_0$ and much longer than the distance to all-burnt, the muzzle energy (which goes as v^2) is proportional to the charge weight, and so to the Force of the powder. However, the muzzle energy will be independent of the projectile weight.

Note that in contrast to the time when the charge is still burning and the velocity is proportional to energy of the propellant gasses (Eqn. 9.9), the velocity at large expansion ratios, well after all-burnt, is proportional to the square root of the energy of the propellant gasses.

§ 9.9 *Pressure as a function of time*

Pressure as a function of time is the most measured ballistic parameter after muzzle velocity, so it would be useful to develop a function for pressure as a function of time. This would appear to be relatively straightforward. Starting on the basis that,

$$v = \frac{dx}{dt} = \frac{dx}{d\left(V/V_0\right)} \frac{d\left(V/V_0\right)}{dt} \qquad 9.39$$

Then, using $V = V_0 + \mathrm{A}x$ and dividing through by V_0,

$$\frac{V}{V_0} = 1 + \frac{\mathrm{A}}{V_0}x \qquad 9.40$$

Differentiating,

$$\frac{d\left(V/V_0\right)}{dx} = \frac{\mathrm{A}}{V_0} \qquad 9.41$$

Substituting into Eqn. 9.39 and integrating,

$$\int_0^t dt' = \frac{V_0}{\mathrm{A}}\int_{V_0}^{V} \frac{d\left(V'/V_0\right)}{v} \qquad 9.42$$

Substituting for v from Eqn. 9.27 the integration can proceed, yielding an expression which gives time as a function of the expansion ratio, from which pressure as a function of time could be readily determined.

$$t = \left[\frac{2V_0}{\mathrm{A}v_K}\left(\frac{V'}{V_0}\right)^{\frac{(\gamma+1)}{2}} \ln\left(1 - \left(\frac{V'}{V_0}\right)^{-\frac{(\gamma-1)}{2}}\right)\right]_{V_0}^{V} \qquad 9.43$$

where,

$$v_K = \left(\frac{4\,g\,V_0\,P_Q}{m_{\mathrm{eff}}}\right)^{\frac{1}{2}} \qquad 9.44$$

It is immediately apparent that there is a singularity at $V' = V_0$ and the resultant equation for time as a function of V/V_0 is not well behaved.

Since all the analytic solutions of the ballistic equations are necessarily of the same general form as those given above, this is a common problem with analytic systems and not just a problem with the Mayer-Hart system. Struble (op. cit.) proved that if it is assumed that the burn rate of the propellant is proportional to the pressure, it is actually not possible to derive a closed analytic form for time. In some systems a function for time is fudged. Corner, for example, avoided the singularity by truncating his solutions so that both velocity and pressure fail to vanish at $t = 0$.

However, as Mayer and Hart commented, if the pressure as a function of distance is known, together with velocity as a function of distance, then determining a pressure-time relationship is, “always possible”. Time as a function of distance x can be plotted from,

$$t(x) = \sum_{x'=0}^{x'=x} \frac{2\,\Delta x'}{v(x') + v(x'+\Delta x')} \qquad 9.45$$

Here, Δx is some small interval in x such that the rate of change in pressure in any interval from x to $(x + \Delta x)$ is approximately linear. Since pressure as a function of distance is known, plotting pressure as a function of time is then straightforward.

§ 9.10 *Corrections*

As noted, a number of approximations were made to facilitate a solution to the ballistic equations. As used originally by the Ballistics Research Laboratory, the only corrections applied were for barrel friction, which was accounted for by increasing the projectile effective mass; and the heat loss which was corrected for by adjusting the ratio of specific heats.

§ 9.10.1 *Accounting for shot-start pressure, recoil and barrel friction*

These matters are discussed in detail in Chapters 5 and 6. The effects of shot-start pressure and barrel friction have the effect of increasing the work done accelerating the projectile up the barrel, so they can be mimicked by increasing the mass of the projectile. The usual amount is 4%. An additional 1% is added to account for the recoil energy of the gun, to give an effective projectile mass of $1.05m$

§ 9.10.2 *Accounting for pressure gradient and kinetic energy of propellant gasses*

These matters are discussed in detail in Chapter 8. By adding a third of the propellant mass to the effective projectile mass above, so that $m_{eff} = 1.05m + C/3$, the kinetic energy of the propellant gasses is accounted for in Résal’s energy balance equation, Eqn. 9.4. The pressure gradient is (approximately) accounted for by using this effective mass in the equation of motion, Eqn. 9.5.

§ 9.10.3 *Accounting for heat loss*

A significant amount of the energy in the gas is lost due to heating up the walls of the chamber and barrel. This is discussed in greater detail in Chapter 7. In analytic systems this is generally accounted for by increasing the value of gamma, which has the effect of decreasing the thermodynamic efficiency of the system and so mimics the heat loss.

A new effective gamma γ_{eff} can be defined where,

$$(\gamma_{eff} - 1) = (\gamma - 1)(1 + k) \qquad 9.46$$

Where k is the ratio of the heat loss to the kinetic energy of the effective mass.

For large guns, heat loss is equivalent to about 30% of the kinetic energy of the projectile, so the effective gamma increases to about 1.3 (where the propellant gamma is about 1.25). For small arms (smaller than .50 calibre), heat loss is equivalent to about 60% of the effective kinetic energy of the projectile, so γ_{eff} is about 1.39. For intermediate 'small cannon', the heat loss can be taken to be about 45% of the effective kinetic energy.

§ 9.10.4 *Using the Mayer-Hart system*

The average vivacities for a number of commercially available small-arms rifle propellant powders are given in Appendix 2. If the Mayer-Hart system is used with these vivacities, and the corrections for barrel friction and heat loss described above, the result stand comparison with those produced by lumped parameter numerical systems and may be considered quite reliable.

§ 9.11 *A worked example*

A worked example is given here for a 30-06 Springfield rifle. The propellant used here was type WC846 which was a double base 'flattened' ball powder, coated in a deterrent to give it a neutral burning characteristic.

Table 9.1 gives a summary of the propellant properties and loading conditions.

Table 9.1 *Propellant properties and loading conditions*

Projectile travel to exit	22.8 inches
Calibre	.308 inches
Bore area A	0.0745 in^2
Projectile weight *m*	150 grains
Charge weight C	50 grains
Case capacity V_C	62.4 grains of water
Propellant burning rate β	2.735 x 10^{-4} in/sec.
Web thickness *w*	0.016 inches
Vivacity Λ	68 /100 bar/sec.
Ratio of specific heats γ	1.225
Propellant Force *F*	3835000 in-lb/lb
Propellant density ρ	0.0584 lb/in^3
Heat loss constant *k*	0.6

§ 9.11.1 *Results*

It will be seen the pressure curve below (Fig. 9.2 and Fig. 9.3) that there is a discontinuity at all-burnt where the propellant stops burning and the adiabatic expansion equation is used thereafter. This discontinuity has never been observed in practice. Due to fracturing and splintering, the propellant kernels disintegrate into fragments which burn at a divergent rate, and the rate of burning peters away to zero in the final stages of burning.

Table 9.2 *Results of Mayer-Hart system compared to experiment.*

	Experiment	**Mayer-Hart**
Maximum chamber pressure	46,876 psi	44,643 psi
Muzzle velocity	2812 ft/sec	2831 ft/sec
Travel to maximum pressure	2 inches	2 inches
Travel to all-burnt	-	9.2 inches

Despite the broad approximations used in the Mayer-Hart system, the calculated trajectory of the projectile up the barrel is in close agreement with that determined in practice.

Due to the assumption of a uniform burning rate, it is not usually expected that the Mayer-Hart system would match the chamber pressure closely, though the agreement in this case is reasonable since the propellant kernel is designed by means of deterrents to burn in a broadly neutral fashion.

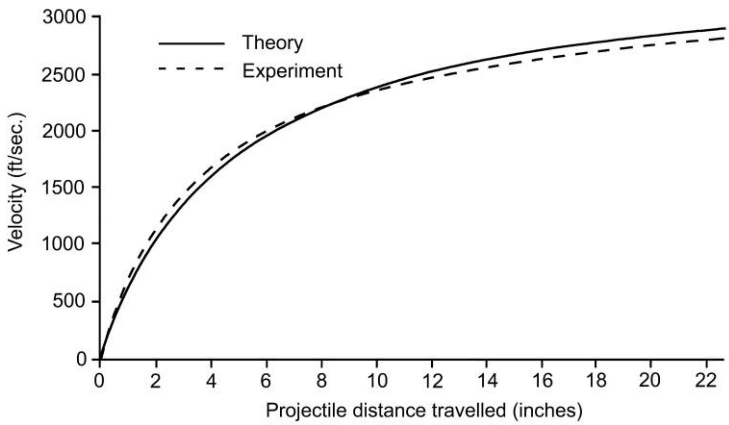

Fig. 9.1 *Velocity as a function of distance travelled – theory and experiment*

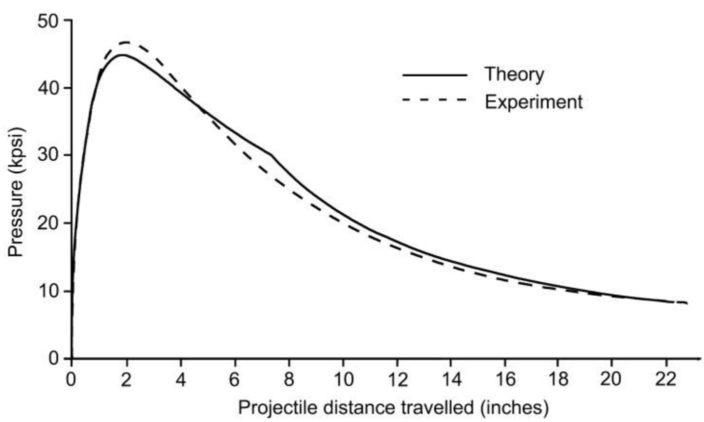

Fig. 9.2 *Pressure against distance – theory and experiment*

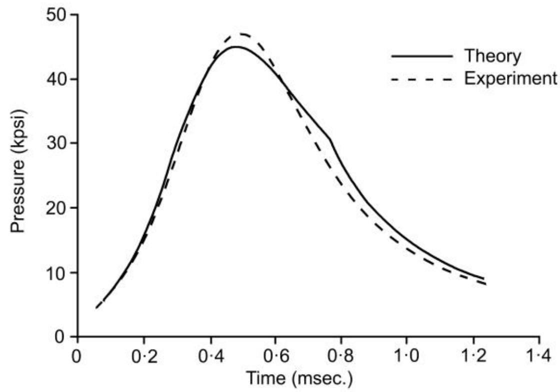

Fig. 9.3 *Pressure against time – theory and experiment*

The plot in Fig. 9.3, was calculated using Eqn. 9.45 from the data in Fig. 9.1 to calculate time as a function of distance, which was then used with the data in Fig. 9.2 to plot pressure against time.

§ 9.11 *Ballistics similitudes*

When designing a new large gun, considerable savings in the costs of research and development are possible by building test guns on a much smaller scale. Suppose a projectile of a given weight was required to be launched from a gun of a certain calibre and length, and at a given muzzle velocity. How could this be tested using a gun scaled down in dimensions by a given factor so that the maximum chamber pressure and muzzle velocity would be the same in the full size gun as they were in the test gun?

The connection between the full size gun and the miniature test gun would be the expansion ratios, which would be the same in each gun at the same stage in the trajectory of the projectile up the barrel.

The barrel length in the test gun would be reduced in proportion to the calibre, and the chamber volume would be reduced in proportion to the cube of the calibre. If it is assumed the same propellant composition is used, the burning rate β and the force F would remain the same. The maximum pressure is proportional to P_Q ,

$$P_Q = \left(\frac{2\,C\,F\,\beta}{A\,w} \right)^2 \frac{m}{g\,V_0}$$

from which it can be seen that the projectile mass must be proportional to initial volume V_0 and so must scale as the cube of the calibre.

From Eqn. 9.37 for the muzzle velocity, it is seen that the ratio of charge weight to projectile weight must remain constant, so the charge weight will also scale as the cube of the calibre.

From the equation for P_Q , the web thickness w must scale as the calibre, since the product of w and the bore area A must scale as the charge weight C. The vivacity of the propellant would then scale as the inverse of the calibre.

Finally, the web thickness must scale as the barrel time for a propellant of a given burning rate, so the time to a given expansion ratio must scale as the calibre.

For example, suppose it is required to find the powder charge, burning rate, web thickness and maximum pressure for a 6" calibre gun, firing a projectile of 100 lb weight at a velocity of 2740 ft/sec. from a barrel of length 50 calibres (300 inches or 25 feet)? The initial chamber volume behind the loaded shell is to be 1000 in^3.

Suppose a test gun is built to a scale of 1:20, which would (conveniently) have a .30 calibre barrel (.300" bore diameter, .308" groove diameter). Barrel length would be 15 inches. The initial usable case capacity would need to be one eighth of a cubic inch, (which is also conveniently the usable case capacity of a 7.62 x 39 cartridge) and the scaled down 30 cal. projectile would weigh $1/80^{th}$ of a pound or 87.5 grains. After some experimentation, it is found that a muzzle velocity of 2740 ft/sec. can be achieved with 0.0376 lb, or 26.3 grains, of a powder with a web thickness of 0.01" and a burning rate of about 4.3×10^{-4} in./sec./psi. Such a propellant would have a vivacity of 123 per 100 bar/sec.

Scaling up, a charge weight of 30 lb of powder having a kernel web thickness of about 0.2" would be required for the 6" gun. The vivacity of the propellant would be one twentieth that required for the miniature test gun, or about 6.15. The maximum pressure and muzzle velocity would be the same as for the test gun.

In fact, the powders used in large guns of this kind actually have burning rates of the order 1.4×10^{-4} in./sec./psi. So to conserve vivacity, the web thickness should be reduced to about 0.06". If the measured barrel time in the test gun was one millisecond, a barrel time for the parent 6" gun should be about 20 milliseconds. These are quite typical parameters for this type of gun.

This principle was used by the Krupp factory in Germany when in the mid 1930's they were called upon to develop (amongst other big guns) an 80cm calibre railway gun capable of smashing the defences of the French Maginot Line – then the heaviest fortifications in existence – in what was to be the Battle of France. "Heavy Gustav" was the largest gun ever used in combat. The gun had a barrel length of 32.5 metres and fired a shell weighing 7 tonnes with a muzzle velocity of 2700 ft/sec. It was finished too late for the Battle of France, but was used with success in other campaigns throughout WWII. To verify the design of the gun, Krupp built a one tenth scale test gun with a calibre of 8cm. Krupp later stated that the propellant and charge so deduced was correct for the full scale gun, without any need for further alteration [7].

§ 9.12 *Approximate systems*

A number of approximate systems have been developed over the years which attempt to give a reasonable estimate for maximum chamber pressure and – in particular – muzzle velocity with the absolute minimum of effort. One example which is still popular amongst European ballisticians is the Vallier-Heydenreich system [13], which was based on empirical parameters derived from shooting a large variety of guns. In the United States the graphical system of Strittmater [14], which was based on the Mayer-Hart solutions above. Homer Powley [15] devised a slide rule that was based on a highly simplified version of Resal's equation, which included some empirical parameters for a number of popular American reloading powders. The system of Leduc [16], was used France, the Soviet Union, and in the United States from the first years of the 20th century right up to the 1970s. Today, the 'Leduc equation' is probably the best known internal ballistics equation and is often cited in the popular media. Accordingly, it will be examined in detail here.

§ 9.13 *The Leduc system*

The Leduc internal ballistics system was first introduced by Challeat [16] in an article on the 'Theory of Recoil' in 1904. The system depends on assuming a simple hyperbolic relation between the projectile velocity and the distance travelled up the barrel, which closely corresponds to the form determined both theoretically and experimentally. The eponymous equation is,

$$v = \frac{a x}{b + x} \qquad 9.47$$

Where '*a*' and '*b*' are constants. The Leduc equation is deceptively simple, which no doubt accounts in large part for its long popularity, but it is never-the-less striking that a whole internal ballistics system can be derived from this simple form.

The derivative of this equation closely corresponds to the form for pressure against projectile travel.

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{a^2bx}{(b+x)^3} \quad 9.48$$

Differentiating again to find the maximum acceleration, where maximum pressure will occur,

$$\frac{d^2v}{dt^2} = \frac{ax(a^2b^2 - 2a^2bx)}{(b+x)^5} = 0 \quad 9.49$$

It follows that at the point of maximum acceleration, $x = b/2$ and so,

$$\left(\frac{dv}{dt}\right)_{\text{P-Max}} = \frac{4\,a^2}{27b} \quad 9.50$$

From Eqn. 9.48 the pressure at any point in the barrel can be written as,

$$P = \frac{m_{\text{eff}}}{\text{Ag}}\frac{dv}{dt} = \frac{m_{\text{eff}}\,a^2bx}{\text{Ag}(b+x)^3} \quad 9.51$$

Where A is the area of the bore and m_{eff} is the effective mass of the projectile, which includes a third of the propellant mass. The acceleration due to gravity *g* is present as pressure here is in units of weight per unit area. From Eqn. 9.51, the maximum pressure can be written as,

$$P_{\text{Max}} = \frac{4\,a^2\,m_{\text{eff}}}{27\,\text{A}\,g\,b} \quad 9.52$$

Rearranging for b,

$$b = \frac{4\,a^2 m_{\text{eff}}}{27\,\text{A}\,g\,P_{\text{Max}}} \quad 9.53$$

If x_{M} is the distance the projectile travels to the muzzle, then from Eqn. 9.47 the constant *a* can be expressed in terms of the muzzle velocity,

$$a = \frac{v_{\text{M}}}{x_{\text{M}}}(b + x_{\text{M}}) \quad 9.54$$

Substituting for a in Eqn. 9.53,

$$b = \frac{4 m_{\text{eff}} v_M^2 (b + x_M)^2}{27 \mathrm{A}\, g\, x_M^2\, P_{\text{Max}}} \qquad 9.55$$

Write,

$$K = \frac{4 m_{\text{eff}} v_M^2}{27\, \mathrm{A}\, g\, x_M^2\, P_{\text{Max}}} \qquad 9.56$$

Then,

$$b = K(b + x_M)^2 \quad \text{or} \quad Kb^2 + (2Kx_M - 1)b + Kx_M^2 = 0 \qquad 9.57$$

Taking roots,

$$b = \frac{1 - 2Kx_M - \sqrt{1 - 4Kx_M}}{2K} \qquad 9.58$$

§ 9.13.1 *A worked example*

Using the example of a 30-06 Springfield rifle given above, The constant K can be determined from Eqn. 9.56 using the values given in Table 1 and Table 2, so that K is 0.0057. From Eqn. 9.58, b is 4.16 inches and from Eqn. 9.54, a is 339900 in/sec. or 3325 ft/sec.

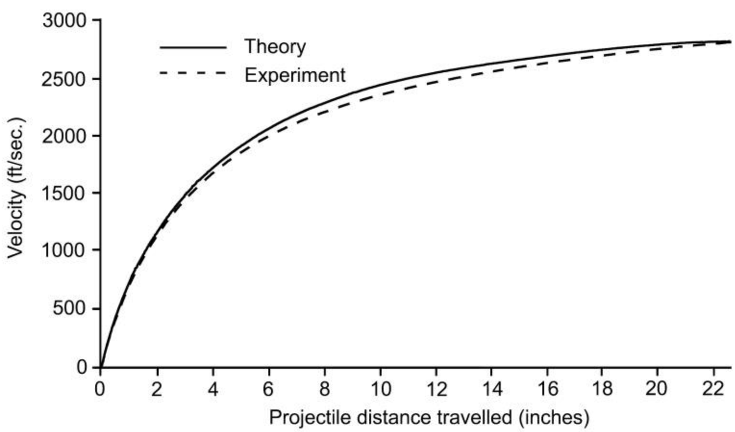

Fig 9.4 *The Leduc curve calculated from maximum pressure and muzzle velocity, compared to the experimental values.*

From these values of a and b it is possible to plot projectile velocity as a function of distance up the barrel using Eqn. 9.47. This is shown in Fig. 9.4 above, with experiment also plotted for comparison.

As can be seen, the agreement with experiment is reasonably good, but as might be expected from such a simple approximation the fit is not exact.

§ 9.13.2 *The Leduc constants in terms of propellant properties and loading conditions*

Serabryakov [17] remarked that needing to know the maximum chamber pressure and the muzzle velocity to determine the Leduc constants, "…reduced their value considerably". Even Leduc, their progenitor, recognised this deficiency in that he did provide forms for a and b which were a function of propellant properties and loading conditions – though they were empirical functions, derived with little support from thermodynamic theory and were tailored to the French propellants of the time. Alger [18] modified these functions for US propellants in 1911 and later, Serabryakov [op. cit.] modified them yet again for Soviet propellants. Despite the evident widespread popularity of the Leduc system, there was always a lingering suspicion regarding these empirical functions for a and b, which were described as "fallacious" by Hunt [19] and not adopted by British ballisticians.

But the use of these empirical functions for so long is surprising, as it is actually quite straightforward to render the Leduc constants in terms of propellant properties and loading condition, based on sound thermodynamic theory.

Assume the approximations set out in § 9.2 above, except that here the covolume of the propellant gasses will here be used explicitly. Also, as above, effective mass is used to account for the pressure gradient in the propellant gasses and for the kinetic energy of propellant gasses. As above too, heat loss is accounted for by using γ_{eff} instead of γ .

§ 9.13.3 *The Leduc constant 'a'*

If a barrel of infinite length is assumed, is it obvious from Eqn. 9.47 that as x, the distance travelled up the barrel, becomes very large, then $v \to a$ and so the constant a is the asymptotic value of the projectile velocity as $x \to \infty$. Following Hunt [op. cit.], Résal's equation is invoked.

$$PV = FZC - (\gamma_{\text{eff}} - 1)\frac{m_{\text{eff}} v^2}{2g} \qquad 9.59$$

At $x = \infty$ the pressure will be zero, Z will be unity and this equation can be rearranged.

$$v = \left[\frac{2FCg}{m_{\text{eff}}(\gamma_{\text{eff}} - 1)} \right]^{\frac{1}{2}} \qquad 9.60$$

In reality of course, barrel friction would ensure that the asymptotic velocity as $x \to \infty$ would actually tend to zero. Let,

$$a = 0.94 \left[\frac{2FCg}{m_{\text{eff}}(\gamma_{\text{eff}} - 1)} \right]^{\frac{1}{2}} \qquad 9.61$$

The empirical factor of 0.94 helps to account for friction in barrels of the usual length and so improves general agreement with experiment.

§ 9.13.4 *The Leduc constant 'b'*

An expression for maximum pressure is given in Eqn. 9.52, but maximum pressure can also be expressed in terms of Résal's equation.

$$P_{\text{Max}} = \frac{FCZ - (\gamma_{\text{eff}} - 1)\dfrac{m_{\text{eff}} v^2}{2g}}{\left(V_0 + \dfrac{bA}{2} \right)} \qquad 9.62$$

The free volume V_0 behind the loaded projectile can be expressed as the initial volume in the chamber behind the loaded projectile V_C , minus the volume of the unburnt powder, minus the volume taken up by the burnt propellant gasses.

$$V_0 = V_C - (1 - Z)\frac{C}{\rho} - ZC\eta \qquad 9.63$$

The term η is the covolume, the volume taken up by a unit mass of the gas molecules themselves.

Now, from Eqn. 9.7, $$v = \frac{Ag}{m_{\text{eff}}} \int P\,dt \qquad 9.64$$

and from Eqn. 9.8, $$Z = \frac{2\beta}{w} \int P\,dt \qquad 9.65$$

Eqn. 9.64 and Eqn. 9.65 can be combined so that,

$$Z = \frac{2\beta m_{\text{eff}} v}{Agw} \tag{9.66}$$

It was shown above that the maximum pressure is reached when $x = b/2$. From Eqn. 9.47 the projectile velocity at maximum pressure is then $a/3$. Substituting for v in Eqn. 9.66 to obtain the value of Z at maximum pressure, and then substituting for Z in Eqn. 9.62 and Eqn. 9.63, Résal's equation can be written as,

$$P_{\text{Max}} = \frac{\dfrac{2FC\beta m_{\text{eff}} a}{3Agw} - (\gamma_{\text{eff}} - 1)\dfrac{m_{\text{eff}} a^2}{18g}}{\left(V_0 + \dfrac{bA}{2}\right)} \tag{9.67}$$

P_{Max} can now be substituted from Eqn. 9.52 and the resultant equation which can be solved for b so that,

$$b = \frac{4aV_0}{\left[\dfrac{18FC\beta}{w} - aA\left(2 + \dfrac{3(\gamma_{\text{eff}} - 1)}{2}\right)\right]} \tag{9.68}$$

Or using vivacity in place of the burning rate,

$$b = \frac{4aV_0}{\left[9FC\Lambda - aA\left(2 + \dfrac{3(\gamma_{\text{eff}} - 1)}{2}\right)\right]} \tag{9.69}$$

The Leduc equations constants a and b have here been determined entirely in terms of propellant properties and loading conditions and can now be used as ballistic coefficients in an internal ballistics system for gas pressure and velocity of the projectile up the barrel.

§ 9.13.5 *All-burnt*

All-burnt occurs when $Z = 1$. The projectile velocity at all-burnt can be determined from Eqn. 9.66 and so the position of all burnt from Eqn. 9.48. Thus, all-burnt happens at,

$$x_{\text{All-Burnt}} = \frac{b}{\left(\dfrac{2\,\beta\, m_{\text{eff}}\, a}{\mathrm{A} g w} - 1\right)} = \frac{b}{\left(\dfrac{\Lambda\, m_{\text{eff}}\, a}{\mathrm{A} g} - 1\right)} \qquad 9.70$$

This is an under-estimation of the position of all burnt in real guns, since propellants do not burn at a constant rate to the end of the burning cycle.

It is usual in analytic systems to assume adiabatic expansion after all-burnt, but since the primary equation for velocity as a function of distance travelled in the barrel is given by the Leduc equation (Eqn. 9.47) there is no need to make such an assumption in this system.

§ 9.13.6 *Using the Leduc system*

Note that in contrast to the Mayer-Hart system, the projectile travel distance to maximum pressure ($b/2$) is weakly dependent on projectile mass (through a), and on propellant burning rate (through β or Λ). In practice however, the Leduc system as given here agrees well with the Mayer-Hart system for normal working pressures (maximum pressures 40 kpsi to 55 kpsi) and gives very similar distances to maximum pressure as the Mayer-Hart system in those circumstances.

For small-arms calibres, the heat loss in this system is assumed to be equal to the projectile energy, so the constant k in Eqn. 9.46 to calculate γ_{eff} should be 1.0 for calibre 0.4" and below. For 0.4" < calibre < 3", k should be 0.5 and for calibres greater than 3", k is 0.3.

§ 9.13.7 *A worked example*

For the 30-06 Springfield example above, using a covolume η for the propellant of 25.454 in^3/lb.

Table 9.3 *Main results for Leduc and experiment*

	Experiment	**Leduc**
Maximum chamber pressure	46,876 psi	46,879 psi
Muzzle velocity	2812 ft/sec.	2886 ft/sec.
Travel to maximum pressure	2 inches	2.4 inches
Travel to all-burnt	-	9.5 inches

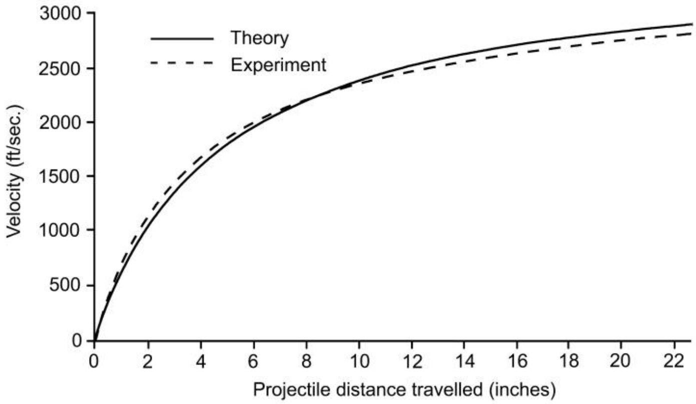

Fig. 9.5 *Velocity as a function of distance, Leduc and experiment*

Here, the theoretical Leduc curve for velocity as a function of distance has been generated using ballistics constants *a* and *b* which were deduced from the propellant properties and loading conditions alone. As can be seen, the agreement is very reasonable.

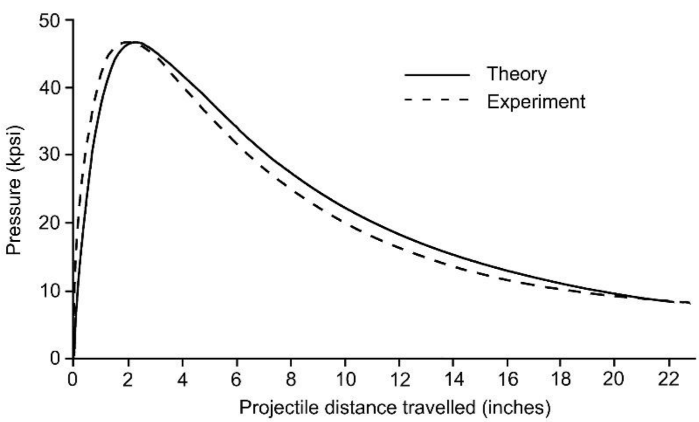

Fig. 9.6 *Pressure as a function of distance, Leduc and experiment*

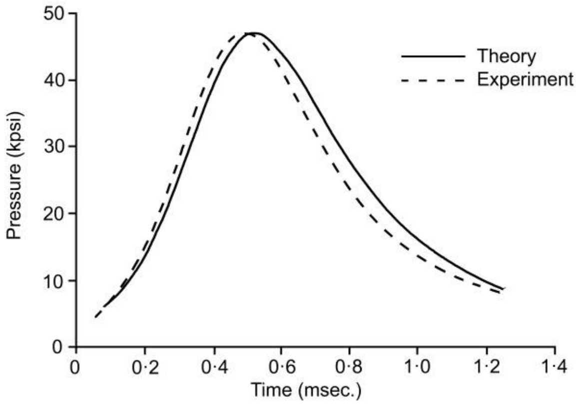

Fig. 9.7 *Pressure as a function of time, Leduc and experiment*

There is no expression for pressure as a function of time in the Leduc system. But as described for the Mayer-Hart system in § 9.9, a plot for pressure against time can be determined from the expressions for pressure and velocity against distance and this is shown in Fig. 9.7.

As with the Mayer-Hart system, the assumption of constant burning rate for the propellant means that the predicted maximum pressures will generally not be very close to those determined in actual firing conditions, but the muzzle velocity should be reasonably reliable. Generally, the Leduc system, using the ballistic coefficients a and b as derived above, gives very reasonable results, considering the simplicity of the system.

§ 9.14 *Forms for black powder*

The work of early ballisticians to develop a mathematical understanding of the internal ballistics of black powder guns was generally disappointing. The decomposition of black powder appears to be extremely complex and very dependent on the conditions in which it is burning, so despite numerous campaigns to determine a burning rate for black powder experimentally, the results have differed widely. In consequence, no analytic form for the internal ballistics of black powder was ever satisfactorily developed, and by the time computing power had become sufficient to attempt a numerical system for black powder, half a century had passed since 'smokeless' powders had overtaken black powder as a propellant of interest so that is where the effort was directed.

However, two investigators have undertaken some interesting work in which a large amount of experimental data for measured muzzle velocities was correlated with the loading parameters (charge weight, projectile weight and barrel length) and simple empirical forms were worked up. The first form given is due to Don Miller [20] where,

$$v = Kl^{1/4}\sqrt{\frac{C}{m + 0.5C}} \qquad 9.71$$

The velocity here is in ft/sec. It is assumed that the projectile is a round ball, so there is no separate term for bore diameter as this is correlated to projectile mass. It was determined that bore size was not otherwise an important factor. The value of the constant K depends on the powder being used and Miller determined the following values for Goex powders.

Table 9.4 *Values of the constant K for various grades of Goex powders*

Grade	**Constant *K***
Fg	1050
FFg	1150
FFFg	1250

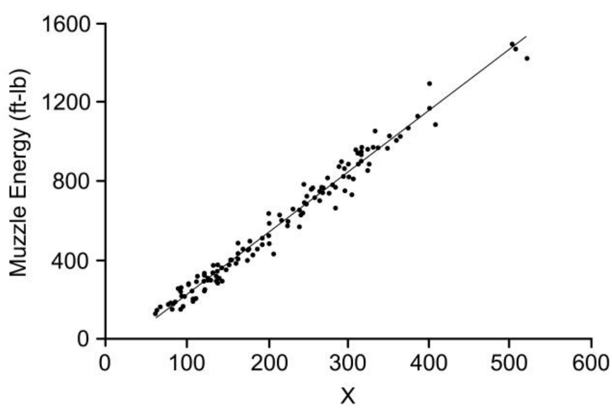

Fig. 9.8 *Muzzle energy as a linear function of 'X' for FFFg powder (with thanks to Henning Umland)*

The second form was independently arrived at by Henning Umland [21] who found a form for muzzle energy that could be correlated with surprising accuracy to a large amount of load data from the Black Powder Loading Manual [22].

Henning Umland deduced that muzzle energy was a linear function of X where,

$$E = -92.67 + 25774X\,, \qquad 9.72$$

where, $$X = (7000^{(0.737+0.282)})C^{0.737} m^{0.282} l^{0.417} = 8282C^{0.737} m^{0.282} l^{0.417}$$

E is the energy in ft. lb. and the barrel length l is in inches. The charge and projectile mass are here in pounds, though in Umland's original equation these masses were in grains (hence the constant 8282 in the equation for X as there are 7000 grains to the lb.) This was for FFFg powder and once again, a round ball is assumed for the projectile.

When X is zero, the muzzle energy has the value of -92.67 ft lb, which at first sight is unexpected. But this will be the energy lost due to friction in the barrel, (averaged, of course, over the numerous barrels for which the data were derived).

It is interesting to convert Miller's equation into one for muzzle energy and compare it to Umland's equation. For FFFg powder ($K = 1250$) then,

$$E = 24269L^{0.5} \frac{Cm}{(m + 0.5C)} \qquad 9.73$$

If it is assumed that the charge weight is significantly less than the projectile mass, then it is easy to see that Eqn. 9.73 approximates remarkably well to Umland's form expressed as Eqn. 9.72,

While useful in their own right for estimating muzzle velocity is black powder guns, these forms are also useful in assessing an equation for the burning rate (regression rate) of black powder to be used in a numerical simulation. By tuning the black powder burning rate constants to fit these forms, which are based in turn on practical measurements, it is possible to use the numerical system more generally for arbitrary projectile weights and shapes, for cartridge guns, and to gain an insight into the pressure history during the ballistic cycle in black powder guns. See § 10.7.

Nomenclature

a = a constant of the Leduc equation in units of velocity: inches/sec.
A = bore area: square inches
b = a constant of the Leduc equation in units of length: inches
C = charge weight: pounds (7000 grains – one lb)
E = muzzle energy: ft lb
F = Force or impetus of the propellant: inch-pounds/inch^3.
g = acceleration due to gravity, which is 386.4 inches/sec^2
l = barrel length: inches
m = mass of the projectile: pounds (7000 grains – one lb)
m_{eff} = the effective mass being accelerated up the barrel: pounds
P = pressure: pounds/inch^2 or psi.
P_B = breech or chamber pressure: pounds/inch^2 or psi.
P_M = mean gas pressure behind the projectile: pounds/inch^2 or psi.
P_{Max} = maximum gas pressure behind the projectile: pounds/inch^2 or psi.
P_S = pressure acting on the base of the projectile: pounds/inch^2 or psi.
t = time: seconds
v = velocity: inches/sec. (Divide by 12 for feet/sec.)
V = volume: inch^3
V_0 = the free space behind the projectile when loaded with propellant: inch^3
V_C = case volume without propellant: inch^3 (252.9 grains of water = one inch^3)
w = the weight of the projectile: pounds (7000 grains = one lb)
x = distance travelled up the barrel by the projectile: inches
Z = the fractional amount of charge burnt: dimensionless
β = the burning rate of the powder: inches/sec/psi.
γ = the ratio of specific heats for the propellant gasses:
γ_{eff} = the ratio of specific heats modified to account for heat loss
ρ = the density of the unburnt propellant: pounds/inch^3
η = the covolume of the propellant gasses: inch^3/pound
Λ = the vivacity of the propellant: /psi/sec. (often quoted /100 bar/sec.)

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