Chapter 10

Numerical systems

§ 10.0 *Introduction*

Numerical solutions to mathematical equations involve the use of numbers in simple arithmetic operations rather than the manipulation of symbols, as is performed for an analytic solution. Numerical solutions are always an approximation to the exact solution, so on this count an analytic solution is always to be preferred – if it is attainable. But there are many occasions where a solution is not available by analytic means, whereas a numerical solution is almost always achievable. More generally, numerical solutions are performed by mechanical means. These days, of course, digital computers are almost universally bent to the task, but prior to dawn of (relatively) cheap, fast, digital computing power in the 1950s, there were electronic analogue computers, which were themselves preceded by 'differential analysers' invented by James Thomson and his brother William Thomson (later Lord Kelvin) in the 1860s and perfected by Vannevar Bush in the early 1930s.

During WWII, Bush differential analysers were used extensively for the solution of internal ballistics problems. By the early 1950's, however, digital computers such the ORDVAC computer, built for the Ballistics Research Laboratory at the Aberdeen Proving Ground in the United States, enabled the internal ballistics equations to be solved quickly, accurately, and without the approximations needed for closed analytical solutions of these equations.

These days, computers have enabled ballisticians to move well beyond the analytic limitations which frustrated them in the mid-20th century, and produce detailed hydrodynamic simulations of internal ballistics processes. But these programs – which can take weeks to run – are principally for answering questions about abnormal performance of guns. The basic equations of classical internal ballistics can give a good understanding of the normal performance of most conventional guns and so these methods are explored here in more detail.

§ 10.1 *Types of numerical internal ballistics systems*

The first reported use of a numerical solution for the internal ballistics equations was that of Baer and Frankle [1]. This code has evolved over the years and is now called the Internal Ballistics High Velocity Gun 2 (IBHVG2) code, which is used extensively in the

United States and by propellant companies in the testing and quality management of their propellants. There are also other standard codes such as used by NATO. These simulations are what is now called 'lumped parameter' systems in that all the propellant in the gun is assumed to burning at the same rate and at the same pressure at any given time. It is also called a 'zero-dimensional' system in that it is assumed that there is no pressure gradient in the gas column behind the projectile, or that the pressure gradient is pre-defined using the Lagrange solution or something similar. The density of the gas is also assumed to be uniform along the length of the gas column.

A 'one-dimensional' system allows the gas density to vary along the axis of the gas column, which is divided into a number of discrete elements having a small length, such that the density, pressure and temperature of the gasses can vary from one element to the next, but are assumed to be constant within the element. This allows the modelling of pressure waves which can propagate within the gas column, reflecting off the projectile base and the breech. Pressure waves arise due to non-uniform ignition and burning of the propellant and can be amplified as they propagate since the burning rate of the propellant is a strong function of the pressure. They can *in extremis* lead to catastrophic destruction of the gun, and so have been a subject of much study since WWII.

'Two-dimensional' systems also allow for variations as a function of radius from the axis, so that any given annulus at some distance 'x' along the gas column and some radius 'r' from the axis of the gas column, will have the same values of density, pressure and temperature. 'Three-dimensional' systems have a set of discrete elements such that variations are also a function of angle around the annulus and are full blown 'finite element' systems.

The goal in any gun is that the propellant should be ignited uniformly and should burn uniformly, so for such systems a lumped parameter system is usually sufficient to describe the ballistics cycle within the gun. Such systems are also very fast such that on a modern computer the virtual run time can be almost as fast as the actual barrel time. A lumped parameter system can even be run on a programmable calculator with a run time of a few minutes.

Only a lumped-parameter zero-dimensional system will be considered here as the more complex systems briefly outlined above are beyond the remit of 'classical' internal ballistics. There are a large number of papers, reports and some books which describe a numerical lumped-parameter system, but usually in such an abbreviated form that they are of little help to the student in creating their own system. An exception is the little

book by Leeming and Farrar [2] which is often cited as the inspiration for a writer's numerical system.

§ 10.2 *The basic elements of a numerical internal ballistics systems*

The main ballistics equations to be solved are set out below. The equations are solved for the time period from t to $t+\Delta t$, where Δt is a short period during which some approximations may be made without significant loss of accuracy.

The first equation gives the amount of gas that has been generated at time $t+\Delta t$ (see Chapter 4).

$$\mathrm{C}Z_{t+\Delta t} = \mathrm{C}Z_t + \mathrm{C}\int_t^{t+\Delta t} \frac{dZ}{dt} dt \qquad 10.1$$

C is the original charge weight. Z is the fractional amount of charge that has burnt at the given time. As discussed in Chapter 4, dZ/dt is a function of pressure, the geometric form of the powder kernel and the rate of regression or 'burning rate' β for the propellant,

$$\mathrm{C}\frac{dZ}{dt} = \mathrm{Area}_Z\, \rho\, \beta\, P^{\alpha} \qquad 4.8$$

For the integral in Eqn. 10.1,

$$\mathrm{C}\int_t^{t+\Delta t} \frac{dZ}{dt} dt = \mathrm{Area}_Z\, \rho\, \beta \int_t^{t+\Delta t} P^{\alpha} dt \qquad 10.2$$

However, if it is assumed that the burning rate is linear with pressure, $(\alpha = 1)$ then it is much more convenient to use the vivacity Λ_Z which is derived from closed bomb tests such that,

$$\frac{dZ}{dt} = \Lambda_Z P \qquad 4.26$$

Then assuming the vivacity does not change significantly over the period Δt,

$$Z_{t+\Delta t} = Z_t + \Lambda_Z \int_t^{t+\Delta t} P_B\, dt \qquad 10.3$$

Where P_B is the breech pressure or chamber pressure in a gun.

The second equation is for the velocity v of the projectile at time $t+\Delta t$. From Newton's second law of motion, the change in velocity of the projectile over the period Δt will be due to the forces on the projectile. Let m be the mass of the projectile and F_S be the force (in pounds) on the projectile base due to the gas pressure then,

$$m\frac{dv}{dt} = g\,F_S \quad \rightarrow \quad dv = \frac{g}{\mathrm{m}}F_S\,dt \tag{10.4}$$

Integrating over the period t to $t+\Delta t$

$$\int_t^{t+\Delta t} dv = \frac{g}{m}\int_t^{t+\Delta t} F_S\,dt \tag{10.5}$$

then,

$$v_{t+\Delta t} = v_t + \frac{g}{m}\int_t^{t+\Delta t} F_S\,dt \tag{10.6}$$

The force F_S can be written in terms of the gas pressure P_S on the projectile base and A, the cross-sectional area of the bore, such that $F_S = \mathrm{A}\,P_S$

Due to the pressure gradient in the gas column behind the projectile (see Chapter 8), the chamber pressure P_B which governs the rate at which the propellant burns can be related to the projectile base pressure by a factor Φ such that $P_S = \Phi\,P_B$

The velocity at time $t+\Delta t$ can now be written as,

$$v_{t+\Delta t} = v_t + \frac{g}{m}\left[\mathrm{A}\,\Phi\int_t^{t+\Delta t} P_B\,dt - \Theta\Delta t\right] \tag{10.7}$$

where Θ is the frictional force retarding the motion of the projectile in the barrel (see Chapter 6) and is usually considered to be a constant.

The third equation is the distance the projectile has travelled up the barrel, which will be,

$$x_{t+\Delta t} = x_t + \int_t^{t+\Delta t} v\,dt \tag{10.8}$$

Finally, the chamber pressure P_B at time $(t+\Delta t)$ can be deduced from Résal's equation (see § 3.5) using the values for $Z_{t+\Delta t}$, $v_{t+\Delta t}$ and $x_{t+\Delta t}$ derived earlier.

$$P_{\mathrm{B},\,t+\Delta t} \;=\; \frac{\mathrm{C}\,\mathrm{Z}_{t+\Delta t}\,F \;-\; (\gamma - 1)\left(\dfrac{m_{\mathrm{eff}}\,v_{t+\Delta t}^{2}}{2\,g} \;+\; E_{h,\,t+\Delta t} \;+\; \Theta\,x_{t+\Delta t}\right)}{\left(V_0 \;+\; \mathrm{A}\,x_{t+\Delta t}\right)} \qquad 10.9$$

Where,

$$V_0 \;=\; V_{\mathrm{C}} \;-\; \frac{\mathrm{C}\left(1 - \mathrm{Z}_{t+\Delta t}\right)}{\rho} \;-\; \mathrm{C}\,\mathrm{Z}_{t+\Delta t}\,\eta \qquad 10.10$$

V_{C} is the original capacity of the empty (no propellant) chamber, or cartridge case, behind the loaded projectile. The density of the propellant (not the bulk density of the propellant powder) is ρ and the co-volume of the propellant gasses (the space taken up by a unit mass of the gas molecules) is η (see § 3.4). The term E_{h} is the heat loss term accounting for heat lost to the walls of the barrel at a given time (see Chapter 7). An effective mass m_{eff} is used here for the projectile energy term which includes a fraction of the mass of the propellant gasses moving up the barrel behind the projectile (see Chapter 8).

By successive iteration through time steps Δt in this way, it is possible to determine the gas pressure, projectile velocity and distance travelled as a function of time during the passage of the projectile up the barrel.

However, it will be appreciated that Equations 10.3 and 10.7 require the integral of pressure P_{B} over time between t and $t + \Delta t$. Since there is no knowledge of P_{B} after time t until Eqn. 10.9, it is necessary to estimate this integral by extrapolation. It is at this point that writers invariably exhort the reader to invoke a generalised forth-order Runge-Kutta procedure for solving ordinary differential equations, which usually comes as a package for various programming languages. Such a sledgehammer is not needed to crack this particular nut though.

In Fig. 10.1, calculated chamber pressure P_{B} from Eqn. 10.9 is plotted for the three time-steps previous to the time step from t to $t + \Delta t$. Assume that over this small time period, these pressure values are well described by a quadratic function of t such that $P_{\mathrm{B}} = at^2 + bt + c$ where a, b and c are constants. The Runge-Kutta procedure assumes that $P_{t+\Delta t} = fn(P, t)$ whereas here it is only assumed that $P_{t+\Delta t} = fn(t)$.

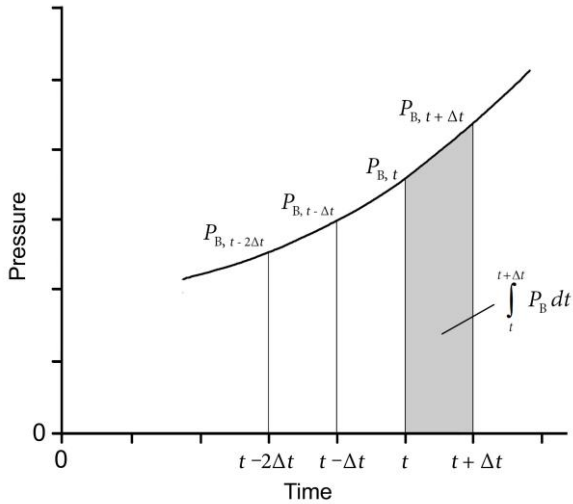

Fig. 10.1 *Plot of chamber pressure as a function of time from Eqn. 10.6*

That being the case, a forth-order Runge-Kutta reduces to Simpson's Rule with equal accuracy [3] and the integral P_B over the period t to $(t+\Delta t)$ can be expressed as (see Appendix 1),

$$\int_{t}^{t+\Delta t} P_B\,dt = \left[5P_{B,\,t-2\Delta t} - 16P_{B,\,t-\Delta t} + 23P_{B,\,t}\right]\frac{\Delta t}{12} \qquad 10.11$$

If it assumed that the burning rate is a function of some power of the pressure α other than unity in Eqn. 10.2, then of course it is possible to write,

$$\int_{t}^{t+\Delta t} P_B{}^{\alpha}\,dt = \left[5P^{\alpha}_{B,\,t-2\Delta t} - 16P^{\alpha}_{B,\,t-\Delta t} + 23P^{\alpha}_{B,\,t}\right]\frac{\Delta t}{12} \qquad 10.12$$

By similar arguments, the integral of velocity over time in Eqn. 10.8 can be expressed as,

$$\int_{t}^{t+\Delta t} v\,dt = \left[5v_{t+\Delta t} + 8v_{t} - v_{t-\Delta t}\right]\frac{\Delta t}{12} \qquad 10.13$$

A flow chart for a numerical system to deliver maximum pressure and muzzle velocity is given in Fig. 10.2. Of course, it is trivial to modify to program to give projectile travel distance to maximum pressure, velocity at maximum pressure, muzzle pressure, muzzle energy and so on.

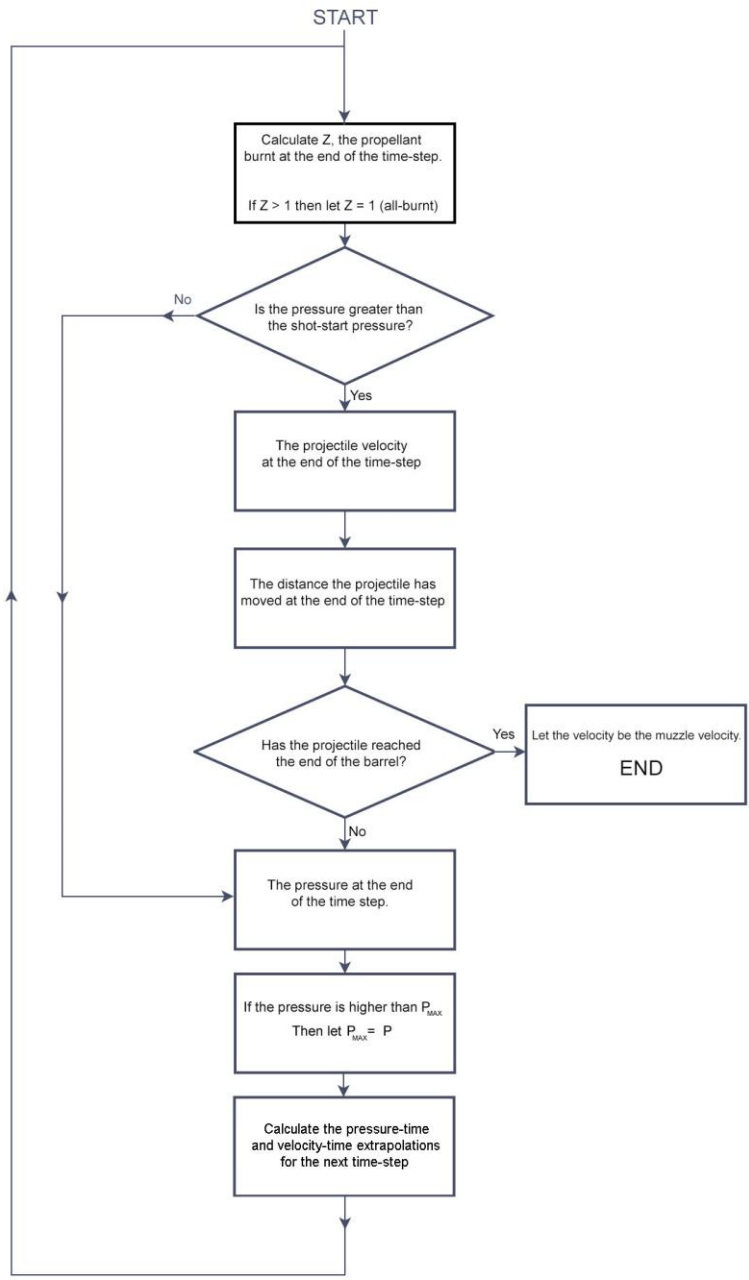

Fig. 10.2 *Flow chart for a practical numerical internal ballistics program*

§ 10.3 *Assumptions usually made*

It is usual to assume that the propellant burning rate is linear with pressure and so the exponent $\alpha = 1$. The engraving force for the projectile into the rifling is usually accounted for by setting a shot-start pressure, below which the projectile does not start moving up the barrel. This is equivalent to seating the projectile up against the lands of the barrel. No work is deemed to be done in engraving the projectile into the rifling, however, since the distance over which the engraving takes place is considered 'small'. Friction as the projectile moves up the barrel is assumed to be constant from the moment the projectile starts to move until it exits the barrel.

It is assumed that the projectile base pressure is constant across the area of the projectile base (no gas friction in the barrel). It is assumed that work done in moving the air ahead of the projectile is not significant. It is assumed the recoil energy of the gun is not significant compared to the projectile energy. Of course, there is little computational difficulty in accounting for these factors, but they are not usually considered significant.

§ 10.4 *Program accuracy as a function of time-step*

The numerical method described can be benchmarked against the Mayer-Hart analytic system (see Chapter 9), if it is set up to have the same initial conditions, where the Mayer-Hart system will give the exact answers. The example here will be for a .30 calibre gun chambered for the 30-06 Springfield cartridge. The required parameters are given in the table below.

Table 10.1 *Propellant properties and loading conditions*

Projectile travel to exit	22.8 inches
Calibre	.308 inches
Bore area A	0.0745 in^2
Projectile weight m	0.0214 lb (150 grains)
Charge weight C	0.0071 lb (50 grains)
Case capacity V_C	0.247 in^3
Vivacity Λ	0.0469
Ratio of specific heats γ	1.224
Heat loss constant k	0.6
Ratio of specific heats γ_{eff}	1.3584
Propellant Force F	3835000 in-lb/lb
Propellant density ρ	0.0584 lb/in^3

The results for the numerical system described above for a number of different values of time-step Δt are given below, and compared to results using the Mayer-Hart analytic system.

Table 10.2 *Comparison of Mayer-Hart and numerical systems for various time-steps*

Δt for numerical system	M-H	10^{-6}	3×10^{-6}	10^{-5}	3×10^{-5}	10^{-4}
Pressure P_{Max} (psi)	44,643	44,643	44,643	44,643	44,629	45,075
Distance to P_{Max} (in.)	2.0	2.0	2.0	2.0	1.9	2.5
Distance to all-burnt (in.)	9.2	9.2	9.2	9.1	9.1	8.6
Muzzle velocity (ft/sec.)	2831	2831	2833	2834	2843	2867
Muzzle pressure (psi)	7716	7716	7681	7645	7487	6988

Total barrel time from when the projectile starts to move to launch from the muzzle is about 1.25×10^{-3} seconds. Where there are about 1000 time-steps $(\Delta t = 10^{-6})$, the numerical system is essentially in exact agreement with the Mayer-Hart analytic solution. For about 100 time-steps $(\Delta t = 10^{-5})$, the discreteness of the time steps start to become apparent in what are essentially rounding errors. But even when there are only about 30 time-steps the error in the maximum pressure is less than 0.2% and the error in the muzzle velocity is only about 0.3%. A time-step of 10^{-4} seconds is of the same order as the rise-time to maximum pressure and the program only cycles through ten time-steps. But even then the error in maximum pressure and muzzle velocity is only about 1%.

The numerical system described is stable and is remarkably invariant to the length of the time-step. But as a general rule, it would seem appropriate that the time-step should be less than one-hundredth of the barrel time.

It should be noted that the Mayer-Hart system switches to the equation of adiabatic expansion after all-burnt, whereas the numerical system above continues to use Résal's equation after all-burnt with the value of Z fixed at one. It was shown in § 3.5.1 that Résal's equation is formally equivalent to the equation of adiabatic expansion if there are zero losses. (Losses for both for the Mayer-Hart system and the numerical system above are not included explicitly, but accounted for implicitly.)

With the accuracy of the numerical system effectively demonstrated, the flexibility of the numerical system can now be utilised by adding in shot-start pressure, barrel

friction, heat loss and pressure gradient as separate variables. The predictive ability of the numerical system then depends on how well those variables are understood.

§ 10.6 *A generic vivacity curve*

For most propellant powders, the vivacity as a function of Z is very similar in form and so little accuracy is lost by assuming a fixed generic form function for all propellant powders. A series of spot values for a generic form function is proposed, which is given below.

Table 10.3 *Spot values for the vivacity form function as a function of* Z

Z	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Form	0.95	0.98	1.00	1.02	1.03	1.03	1.02	0.96	0.80	0.50	0.05

It is convenient to fit polynomial functions to these spot values for the vivacity where,

For $0 < Z < 0.6$, $\Lambda_Z = \Lambda_{AV}\left(0.950 + 0.294\,Z - 0.131\,Z^2 - 0.278\,Z^3\right)$ Curve 'A'

For $0.6 \leq Z < 1$, $\Lambda_Z = \Lambda_{AV}\left(0.532 + 0.256Z + 3.430\,Z^2 - 4.170\,Z^3\right)$ Curve 'B'

Here, Λ_{AV} is an 'average' vivacity, which is determined as though the burn rate was constant with Z and which is listed for a number of propellant powders in Appendix 2.

It should be noted that vivacity form functions as devised by Charbonnier (see Chapter 4) usually have a value of unity at Z = 0 and the vivacity is defined as the initial rate at which the volume decreases (per unit of pressure) as the propellant starts to burn. Here, an average vivacity has been defined, which would be the vivacity of a propellant powder for which the burn rate was constant (invariant with Z). The generic form function thus starts a little below unity, and is progressive in nature up to about Z = 0.5, after which it becomes more strongly digressive to reflect splintering of the propellant kernels. The advantage of this approach is that the average vivacity so defined can also be used in the analytic models described in Chapter 9.

In practice, the vivacity is zero at Z = 1 as burning stops at that point. But when determining the point at which 'all-burnt' occurs, a numerical model might never reach all burnt as there would always be some small residual of propellant left where the vivacity is tending to zero when the projectile leaves the barrel. So that realistic values of the position of all-burnt can be determined then, the vivacity form function is set at 0.05 for Z = 1.

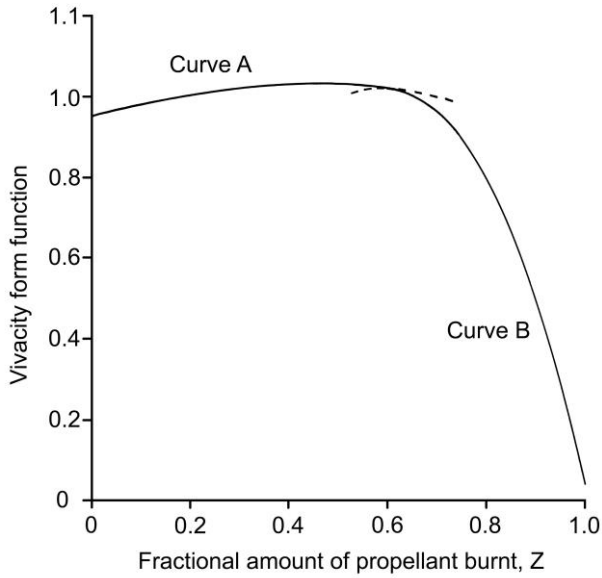

Fig 10.2 *A generic vivacity curve for small arms propellant powders. Two third order polynomials are fitted to a series of proposed spot values for the form function, meeting at* Z = 0.6

§ 10.7 *A numerical model for black powder*

There appears to have been very little work done to develop numerical systems for black powder propellants. This is not surprising, as by the time computing power had developed to the point where it was feasible to model internal ballistics systems numerically, 'smokeless' nitrocellulose propellants had superseded black powder for half a century. In military use, black powder retains a place only in the initiators of the propellant charge in large guns. But it is still of interest to model the performance of historic black powder guns, (see, for example [4]) and there is a popular interest in reproduction black powder small-arms, so that a numerical system for black powder guns is useful.

As related in Chapter 4, the burning process of black powder is complex and it has been found difficult to determine a consistent and accurate value for the burning rate of black powder. The burning rate would also appear to be sensitive to the exact composition of the charcoal used in the compound. Since it is a natural substance, the composition of charcoal is actually quite complex.

However, if it is assumed that the black powder kernels are spherical with radius r, then the following regression rate form would appear to work reasonably well in a numerical system as set out above, to reproduce measured muzzle velocities and estimated pressures in black powder guns.

$$\frac{dr}{dt} = 0.65\,P^{0.3} \qquad 10.14$$

The rate of regression dr/dt of the surface is in inches per second and P is the pressure in psi. This form is similar to regression rate forms previously determined by Sassé [5]. An equation for Z can then be written as,

$$\mathrm{Z}_{t+\Delta t} = \mathrm{Z}_t + \frac{r_t^3 - r_{t+\Delta t}^3}{r_{t=0}^3} \qquad 10.15$$

where,

$$r_{t+\Delta t} = r_t - 0.65 \int_t^{t+\Delta t} P_{\mathrm{B}}^{0.3}\,dt \qquad 10.16$$

The radius $r_{t=0}$ is half the diameter of the unburnt kernel.

The 'quickness' of the powder will depend on the diameter of the kernel. Coarse grained powders are used for cannons and fine grained powders are used for small arms. Table 10.4 gives a list of kernel diameters for different powder grades.

Table 10.4 *Mesh size and average kernel diameters of various black powder grades*

Powder grade	Mesh size	Diameter (in.)
Whaling	4	0.187
Cannon	6	0.132
Saluting	10	0.079
Fg	12	0.066
FFg	16	0.047
FFFg	20	0.033
FFFFg	40	0.019
FFFFFg	75	0.006

Table 10.5 *Black powder propellant properties*

Ratio of specific heats γ	1.22
Co-volume η	18.2 in^3/lb
Propellant Force F	1200000 in-lb/lb
Propellant density ρ	0.0506 lb/in^3
Bulk density	0.035 lb/in^3 (approx.)

Other parameters for black powder propellant (see [6]) are listed in Table 10.5.

A numerical system as set out above gives a good match to the empirical forms for muzzle velocity given by Don Miller [7] and for muzzle energy by Henning Umland [8] for black powder small-arms (see Chapter 9).

Nomenclature

A = bore area: square inches
C = charge weight: pounds (7000 grains – one lb)
E = muzzle energy: ft lb
E_h = Heat loss to the barrel: inch-pounds
F = Force or impetus of the propellant: inch-pounds/inch^3.
F_S = the force on the projectile base: pounds
g = acceleration due to gravity, which is 386.4 inches/sec^2
l = barrel length: inches
m = mass of the projectile: pounds (7000 grains – one lb)
m_{eff} = the effective mass being accelerated up the barrel: pounds
P = pressure: pounds/inch^2 or psi.
P_B = breech or chamber pressure: pounds/inch^2 or psi.
P_M = mean gas pressure behind the projectile: pounds/inch^2 or psi.
P_{Max} = maximum gas pressure behind the projectile: pounds/inch^2 or psi.
P_S = pressure acting on the base of the projectile: pounds/inch^2 or psi.
t = time: seconds
Δt = small interval of time: seconds
v = velocity: inches/sec. (Divide by 12 for feet/sec.)
V = volume: inch^3
V_0 = the free space behind the projectile when loaded with propellant: inch^3
V_C = case volume without propellant: inch^3 (252.9 grains of water = one inch^3)
x = distance travelled up the barrel by the projectile: inches
Z = the fractional amount of charge burnt: dimensionless
β = the burning rate of the powder: inches/sec/psi.
γ = the ratio of specific heats for the propellant gasses:
γ_{eff} = the ratio of specific heats modified to account for heat loss
ρ = the density of the unburnt propellant: pounds/inch^3
η = the covolume of the propellant gasses: inch^3/pound
Λ_Z = the vivacity of the propellant: /psi/sec. (often quoted /100 bar/sec.)
Θ = projectile frictional force in the barrel: inch-pounds

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