Chapter 8

Pressure Gradients

§ 8.0 *Introduction*

For the propellant gasses to follow the projectile down the barrel and continue to accelerate the projectile, there needs to be some force acting on the propellant gasses. This force is present due to a pressure gradient in the propellant gasses, such that the pressure is lowest at the projectile base and highest at the breech end or in the chamber of the barrel. Given that the pressure acting on the projectile is then not the same as the pressure affecting the burning rate of the propellant, some account needs to be given for the difference in these pressures.

This problem was first addressed at the end of the 18th century by Lagrange [1], the Italian/French mathematician, who gave a solution based on hydrodynamic considerations. Various 'improvements' and alternatives to the Lagrange solution have been put forward over the years, but experimental evidence has shown it to be essentially correct in conventional propellant guns where the ratio of charge to projectile weight is less than one. The only correction that is obviously needed to the Lagrange solution is for that period of the ballistic cycle between the projectile starting to move up the barrel and approximately the point of maximum pressure and this is discussed below.

§ 8.1 *The Lagrange velocity gradient*

A requirement for a solution to the pressure gradient in the propellant gasses behind the projectile is a knowledge of the velocity gradient of those gasses.

Consider a gun in which the projectile has moved a distance x = L down a barrel with a

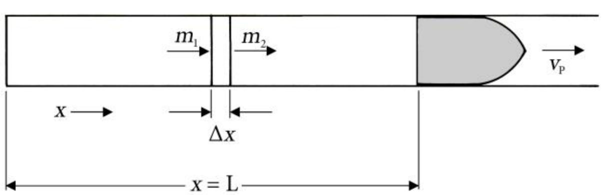

Fig. 8.1 *Element Δx of the bore of the barrel with gas flowing into and out of it*

bore of cross sectional area A. The projectile has a weight W and its velocity at this point is v_P. Consider an element of width Δx at some point behind the projectile. See Fig. 8.1.

Let the mass of gas flowing into the element in a given period be m_1 and that flowing out be m_2 . The rate of change in the mass of gas within the element will be dm/dt such that,

$$\frac{dm_2}{dt} = \frac{dm_1}{dt} - \frac{dm}{dt} \qquad 8.1$$

Let ρ_1 and v_1 be the density and velocity of the gas flowing into the element, so that,

$$\frac{dm_1}{dt} = \mathrm{A}\rho_1 v_1$$

Then,

$$\mathrm{A}\rho_2 v_2 = \mathrm{A}\rho_1 v_1 - \mathrm{A}\Delta x \frac{d\rho}{dt} \qquad 8.2$$

The differences in densities and velocities can be expressed in terms of a gradient with x so that,

$$\mathrm{A}\left(\rho_2 v_2 - \rho_1 v_1\right) \rightarrow \mathrm{A}\Delta x \frac{d}{dx}\left(\rho v\right) = -\mathrm{A}\Delta x \frac{d\rho}{dt} \qquad 8.3$$

Differentiating the left hand side by parts and simplifying,

$$\rho \frac{dv}{dx} + v\frac{d\rho}{dx} = -\frac{d\rho}{dt} \qquad 8.4$$

The term $d\rho/dx$ can be set to zero because the gas is deemed to be ‘incompressible’ in these circumstances. The term incompressible is strange because, of course, a gas is compressible. But the meaning in fluid dynamics is that any attempt to compress the gas – and so raise its density – at some local point in the gas will be countered by the gas flowing such that an equilibrium in density is restored. This assumes that the gas can flow quickly enough in the timescales of interest. Eqn. 8.4 then reduces to,

$$\frac{dv}{dx} = -\frac{1}{\rho}\frac{d\rho}{dt} \qquad 8.5$$

This equation is called the *continuity equation* of the gas. Let C be the propellant charge weight and Z be the fraction of propellant that has burnt. The mass of propellant gas in the barrel behind the projectile is then CZ at any given time, and the density of gas can be written as $\rho = \mathrm{CZ}/\mathrm{AL}$.

The differential for density on the right hand side of Eqn. 8.5 can then be written as,

$$\frac{d\rho}{dt} = \frac{C}{A}\frac{d}{dt}\left(\frac{Z}{L}\right) = \frac{C}{A}\left(\frac{L\dfrac{dZ}{dt} - Z\dfrac{dL}{dt}}{L^2}\right) \qquad 8.6$$

This expression can be conveniently simplified if the propellant is considered to be all-burnt, then Z = 1 and dZ/dt can be set to zero.

The term dL/dt is just the velocity of the projectile at L. Eqn. 8.5 can then be written as,

$$\frac{dv}{dx} = \frac{v_P}{L} \quad \text{or} \quad dv = \frac{v_P}{L}dx \qquad 8.7$$

Integrating,

$$v = \frac{v_P}{L}x + \text{Constant} \qquad 8.8$$

When $x = 0$, the velocity is necessarily zero and so the Constant is zero. Finally then,

$$v = \frac{v_P}{L}x \qquad 8.9$$

The velocity of the gas behind the projectile thus increases linearly from zero at $x = 0$, to v_P at $x = L$. However, it should be noted that this relationship is only shown to be true when the propellant is all-burnt. Many writers assume this relationship also holds while the propellant is still burning, but it is easy to show (see below) that this cannot be assumed at all times before all-burnt.

§ 8.2 *The Lagrange pressure gradient*

Consider now the pressures on either side of the element of the gas.

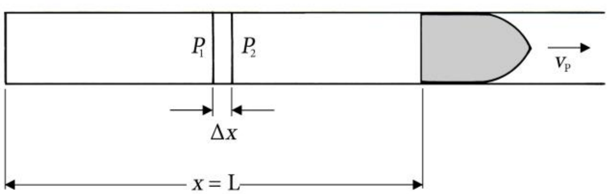

Fig. 8.2 *Pressure of the propellant gasses on either side of an element of width Δx*

The force on the element of gas accelerating it up the barrel will be due to the difference in pressure *P* on either side of the element.

$$\mathrm{A}gP_1 = \mathrm{A}gP_2 + \frac{d}{dt}(vm) \rightarrow \mathrm{A}g\frac{dP}{dx}\Delta x = -\frac{d}{dt}(vm) \qquad 8.10$$

where m is the mass of gas in the element and v is its velocity. This is just an expression of Newton's second law of motion, that force (pressure times area) equals the rate of change of momentum.

It is convenient to introduce a dimensionless variable s, such that $s = x/\mathrm{L}$. So $\Delta x = \mathrm{L}\,\Delta s$ and when $x = \mathrm{L}$, then $s = 1$.

From Eqn. 8.9, $v = s v_P$ and $m = \mathrm{C\,Z}\Delta s$ on the basis that gas density is constant along the length of the gas column. Since s and Δs are not time dependant, Eqn. 8.10 can now be rewritten as,

$$\mathrm{A}g\frac{dP}{ds}\Delta s = -\mathrm{C}s\Delta s\frac{d}{dt}(v_\mathrm{P}\mathrm{Z}) \qquad 8.11$$

Integrating forward from the element to the base of the projectile,

$$\mathrm{A}\int_{P}^{P_S} dP = -\frac{\mathrm{C}}{g}\frac{d}{dt}(v_\mathrm{P}\mathrm{Z})\int_{s}^{1} s'ds' \qquad 8.12$$

then,

$$(P - P_\mathrm{S}) = \frac{\mathrm{C}\,(1 - s^2)}{2\mathrm{A}g}\left(v_\mathrm{P}\frac{d\mathrm{Z}}{dt} + \mathrm{Z}\frac{dv_\mathrm{P}}{dt}\right) \qquad 8.13$$

The term dv_P/dt is just the acceleration of the projectile and this can be equated to the force on the projectile base, which is the pressure P_S on the projectile base, times the area A, divided by the weight W of the projectile. (Bore friction is ignored here.)

$$\frac{dv_P}{dt} = \frac{\mathrm{A}gP_\mathrm{S}}{\mathrm{W}} \qquad 8.14$$

The pressure at the back or breech of the gun, where $s = 0$, will be P_B and so the projectile base pressure P_S can be written in terms of the breech pressure from Eqn. 8.13

$$(P_\mathrm{B} - P_\mathrm{S}) = \frac{\mathrm{C}}{2\mathrm{A}g}\left(v_\mathrm{P}\frac{d\mathrm{Z}}{dt} + \frac{\mathrm{A}gP_\mathrm{S}\mathrm{Z}}{\mathrm{W}}\right) \rightarrow \frac{v_\mathrm{P}\mathrm{C}}{2\mathrm{A}g}\frac{d\mathrm{Z}}{dt} + \frac{\mathrm{C}P_\mathrm{S}\mathrm{Z}}{2\mathrm{W}} \qquad 8.15$$

§ 8.2.1 *The Lagrange pressure gradient after all-burnt*

After all-burnt, when Z = 1 and $dZ/dt = 0$ then Eqn. 8.15 simplifies to the classic Lagrange pressure gradient form,

$$P_B = P_S\left(1 + \frac{C}{2W}\right) \qquad 8.16$$

Similarly, after all-burnt, from Eqn. 8.13, the pressure P as a function of s can be expressed as,

$$P = P_S + (P_B - P_S)(1 - s^2) \qquad 8.17$$

The mean pressure $\overline{P}$ can be determined by integrating for pressure as a function of distance s along the gas column, and dividing by the length of the gas column (which in this case is $s = 1$).

$$\overline{P} = \int_0^1 P ds = \int_0^1 \left(P_B + P_S s^2 - P_B s^2\right) ds \rightarrow P_S\left(1 + \frac{C}{3W}\right) \qquad 8.18$$

Or, substituting for P_S from Eqn. 8.16,

$$\overline{P} = P_B \frac{\left(1 + \frac{C}{3W}\right)}{\left(1 + \frac{C}{2W}\right)} \qquad 8.19$$

For analytic solutions to the ballistics equations it was generally assumed that the all-burnt Lagrange pressure gradients apply from the moment the projectile starts to move. This might have been a reasonable assumption in the days of black power, but it is certainly not the case for modern nitrocellulose propellants, where typically the projectile will start to move where only about 2% of the propellant has burnt.

To circumvent this problem, many writers assume that the propellant gas and unburnt propellant kernels can be lumped together as an homogeneous fluid. The pressure at which the propellant is considered to burn is then the mean gas pressure $\overline{P}$ on the basis that the unburnt propellant will be spread evenly throughout the gas column. However, another reasonable scenario is that the propellant does not move but remains static in the chamber, and it is only the propellant gasses that move up the barrel behind the projectile. The truth would appear to be between these two extremes.

Corner [2] gives an analysis of the motion of a variety of propellant shapes in the barrel behind the projectile due to the pressure differences on the ends of the kernel. He concluded that the movement of the propellant kernels was only of the order of a few calibres before the kernel was entirely burnt. This analysis is broadly in agreement with the results of a modern computational fluid dynamics model [3] and also with an experiment from 1935 [4] in which radioactive particles embedded in propellant kernels were tracked down the barrel of a 37mm gun (See [5] for a description of this experiment.)

Corner's conclusion that, "…the rate of burning at any time is more nearly controlled by the pressure in the chamber than by the mean pressure in whole volume behind the shot…", would appear to be reasonable.

Accordingly, is will be generally assumed in this book that the propellant remains stationary while it is burning and only the propellant gasses follow the projectile down the barrel.

§ 8.2.1 *The Lagrange pressure gradient before all-burnt*

The term dZ/dt in Eqn. 8.15 can be estimated generally from Eqn. 4.9 which is often expressed in terms of a web thickness *w*, as here for neutral burning propellants,

$$\frac{dZ}{dt} = \frac{\text{Area}_Z \, \beta \, P_B^{\alpha}}{\text{Volume}} \rightarrow \frac{2\,\beta\, P_B^{\alpha}}{w} \qquad 8.20$$

Or dZ/dt can be expressed in terms of vivacity Λ_Z which is a function of Z (see Chapter 4) such that,

$$\frac{dZ}{dt} = \Lambda_Z \, P_B \qquad 4.26$$

Then Eqn. 8.15 can be expressed as,

$$P_S = \frac{P_B\left[1 - \dfrac{C v_P \Lambda_Z}{2 A g}\right]}{\left(1 + \dfrac{CZ}{2W}\right)} \qquad 8.21$$

Note that now, the pressure ratio is unity when the projectile start to move, that is when $v_P \simeq 0$ and $Z \simeq 0$, which is what has shown to be the case experimentally. In a numerical system, all variables will be known at any given time and so the pressure ratio as expressed in Eqn. 8.21 can be computed as a running variable during the ballistic cycle.

This approach, when Z < 1, was first mooted by Thornhill [6] and taken up by various writers thereafter (see Gupta [7]).

It is assumed that the velocity gradient in the gas column behind the projectile is linear, ranging from zero at the breech end to v_P behind the projectile. Thornhill thought that this is "clearly" the case (ibid), but it is worth examining the extent to which it is true and how much effect not being true would have on the argument set out above.

In general, it is only possible to determine the velocity gradient while the propellant is burning by use of a full computational hydrodynamic model. However, there is a particular moment where it is possible to say what the gas density is as a function of time, and what the gas velocity is as a function of distance down the gas column. That is the moment of maximum pressure when the rate at which pressure is changing with time is zero. Re-writing the ideal gas equation in terms of density,

$$P = \rho \frac{RT}{M}$$

Differentiating with time,

$$\frac{dP}{dt} = \frac{d\rho}{dt}\frac{RT}{M} = 0 \qquad 8.22$$

If the rate of change of pressure is zero, then the rate of change of density is zero. And from the continuity equation,

$$\frac{dv}{dx} = -\frac{1}{\rho}\frac{d\rho}{dt} = 0 \qquad 8.23$$

the rate of change of velocity with distance must also be zero and there is no velocity gradient in the gas column.

Fig. 8.3 *At the moment of maximum pressure, the burning propellant is generating gasses at the same rate as the volume is increasing. The whole gas column thus has the same velocity as the projectile.*

Physically, this can be appreciated as the rate of production of gas by the burning propellant is being matched by the rate of increase in volume as the projectile moves down the barrel. To first approximation then, the entire propellant gas column will be moving down the barrel at the projectile velocity.

Since the projectile is still accelerating, assume the back of the gas column is accelerating at the same rate as the front of the gas column. Then,

$$\frac{dv}{dt} = A\frac{P_S}{W} = A\frac{P_B}{\left(CZ + W\right)} \qquad 8.24$$

and,

$$P_S = \frac{P_B}{\left(1 + \dfrac{CZ}{W}\right)} \qquad 8.25$$

The validity of Eqn. 8.21 for all Z is then doubtful. It has also been shown that experimentally, Eqn. 8.21 generally under-estimates the projectile base pressure when Z < 1.

Prior to the propellant being all-burnt, most writers assume that the propellant gas and unburnt propellant kernels can be lumped together as an homogeneous fluid, so that the Lagrange solutions are also valid when the propellant is burning. It is not correct to make this assumption. Other significant effects such as bore friction, gas friction on the barrel walls and chambrage (the effect of the gun chamber in holding back the propellant) are also not taken into account in the Lagrange solution.

A pragmatic approach, which fits reasonably well with experimental results, is to assume Eqn. 8.25 is true for $Z \leq {}^1/_2$ after which the pressure ratio reverts to Eqn. 8.16, the classic Lagrange pressure ratio.

§ 8.3 *The kinetic energy of the propellant gasses*

In addition to the work being done accelerating the projectile up the barrel, work is also being done to accelerate the propellant gasses up the barrel and this needs to be accounted for.

The kinetic energy of the element of length Δs (see Fig. 8.2) will be ΔE where,

$$\Delta E = \frac{m v^2}{2g} \qquad 8.26$$

The mass of the element is m and as above, this can be replaced by $CZ\Delta s$ on the assumption that the density of the gas is uniform along the gas column. Assuming the velocity gradient of the propellant gasses behind the projectile is linear, then the velocity

v can be replaced by $s v_P$ and the energy of the gas element can be expressed as a function of s. Let the width of the element $\Delta s \rightarrow 0$ then,

$$dE = \frac{C Z v_P^2 s^2}{2g} ds \qquad 8.27$$

Integrating over the length of the gas column,

$$E = \frac{C Z v_P^2}{2g} \int_0^1 s^2 ds \rightarrow \left(\frac{C Z}{3} \right) \frac{v_P^2}{2g} \qquad 8.28$$

It may be considered, then, that the kinetic energy of the propellant gasses at any moment is equivalent to that of a third of the mass of the gasses travelling at the same velocity as the projectile.

§ 8.5 *The validity of the Lagrange pressure gradient*

Love [8] showed that the movement of projectile in the barrel should give rise to a rarefaction wave which then reflects back and forth between the breech and the projectile base. Pidduck [8] gave a limiting solution to Love's equations assuming that the rarefactions waves were completely damped. Kent [9] reported an error in Pidduck's method and re-derived the limiting solution in a different way to give what is now called the Pidduck-Kent solution.

Where the propellant charge weight starts to become comparable with, or even larger than the projectile weight, Kent and Pidduck showed that a correction needs to be made to the Lagrange solution. Many numerical computer systems give a choice between Lagrange or Pidduck-Kent for the pressure gradient. Kent showed that,

$$\frac{P_B}{P_S} = 1 + \frac{1}{2}\left(\frac{C}{W}\right) - \frac{1}{24\gamma}\left(\frac{C}{W}\right)^2 + \frac{1}{10\gamma}\left(\frac{1}{8} + \frac{1}{36\gamma}\right)\left(\frac{C}{W}\right)^3 + \cdots \qquad 8.29$$

And the kinetic energy of the propellant gasses will be,

$$E = \frac{C v_P^2}{6}\left[1 - \frac{2}{15\gamma}\left(\frac{C}{W}\right) + \frac{(24\gamma + 8)}{630\gamma^2}\left(\frac{C}{W}\right)^2 + \cdots\right] \qquad 8.30$$

The term γ is the ratio of specific heats for the propellant gasses.

Kent noted that the Lagrange equations above would be accurate to better than one percent provided the propellant charge weight was less than half the projectile weight.

This is usually the case in most practical situations, though for modern tank ammunition $C/W \approx 1$ and the error in Eqn. 8.16 would be of the order of 2%.

It was assumed for the Pidduck-Kent solution, that the propellant was all-burnt. In consequence, these solutions for the pressure gradient and gas kinetic energy above are only strictly valid when the propellant is all-burnt and there is no change in the mass of gas in the barrel.

Experimental evidence shows that the ratio of projectile base pressure to chamber pressure is in good agreement with the Lagrange solution after the charge is all-burnt, particularly where the charge weight is small compared to the projectile weight.

§ 8.7 *Experimental measurements of pressure ratios*

There appear to have been comparatively few campaigns described in the open literature where experimental results are presented for the ratio between the projectile base pressure and the chamber pressure.

The earliest of such descriptions is the work of Goode and Lockett in 1944 [10] which is also described in Thornhill [ibid]. The experimental method was to place pressure transducers in ports at a number of places down the barrel of a 6" naval gun of 50 calibre length, and measure the pressure as the projectile base passed the port. A typical result would be as shown in Fig. 8.4.

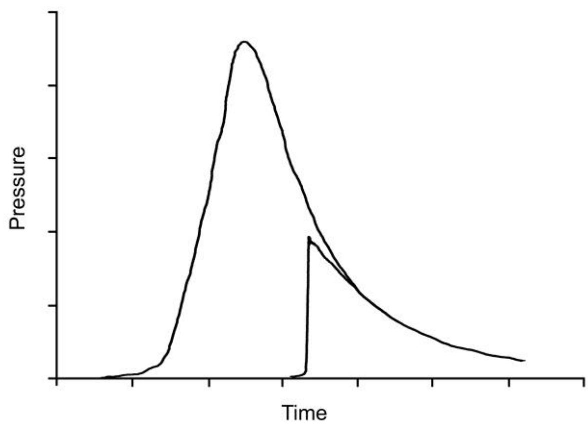

Fig 8.4 *Pressure as a function of time as measured by a piezo transducer in the chamber, and a transducer placed at some distance down the barrel.*

As the projectile passes a down-barrel transducer. The pressure rises very quickly (with a slight overshoot) to the projectile base pressure. Thereafter, the pressure at that point gradually approaches the chamber pressure as its position on the pressure gradient to the projectile base moves relatively further back.

Two charge weights of powder were used by Goode and Lockett so that two different Lagrange ratios for P_S/P_B could be tested. The results for the 31 lb charge in the experiment of Goode and Lockett are shown in Fig. 8.5 and are compared to the predictions of Eqn. 8.25 when $Z \le {}^1\!/_2$ and Eqn. 8.16 thereafter.

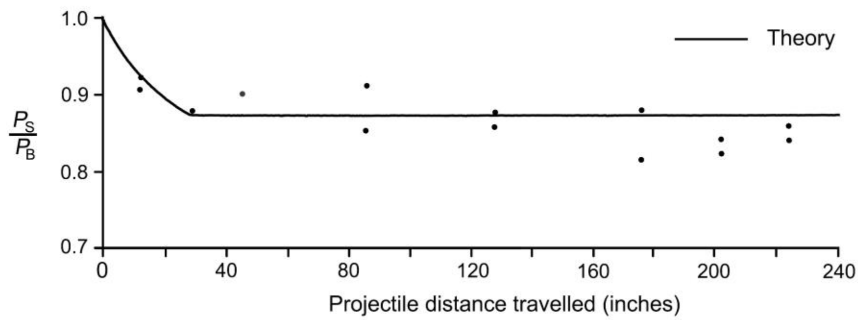

Fig. 8.5 *Projectile base to chamber pressure ratio as a function of distance travelled in a 6 inch BL naval gun for 31 lb propellant charge*

An estimation of Z for Eqn. 8.25 was made using the expression for Z as a function of expansion ratio in the Mayer-Hart system, Eqn. 9.24. (The pressure gradient is accounted for in the Mayer-Hart system by use of an effective mass, which is invariant with projectile distance travelled, but the accuracy is reasonable for this purpose.)

Table 8.1 *Parameters used in calculating Z for 31lb charge in the 6" BL naval gun*

Force of Cordite SC	$4.4\text{x}10^6$ in-lb/lb
Ratio of specific heats for Cordite SC γ	1.248
Charge weight of Cordite SC 150	31 lb
Projectile weight	100 lb
Estimated Propellant vivacity Λ	6.3 /100 bar/sec.
Chamber capacity behind projectile V_C	1500 in^3
Maximum chamber pressure (given)	44550 psi
Muzzle velocity (given)	2850 ft/sec.
Heat loss constant used k	0.17

The vivacity of the powder charge was adjusted so that the predicted maximum pressure and muzzle velocity approximated those reported by Goode and Lockett. Other parameters used are given in Table 8.1.

The ratio of charge weight to projectile weight here is 0.31, so the Lagrange ratio of pressures P_S/P_B is 0.87. The measured pressure ratios are a reasonable match to Eqn. 8.25. The measured ratio is lower than theory towards the muzzle, but it is reported that the test gun was "rather more than half worn" and muzzle erosion would have caused significant leakage around the projectile, which would have lowered the effective base pressure in this region.

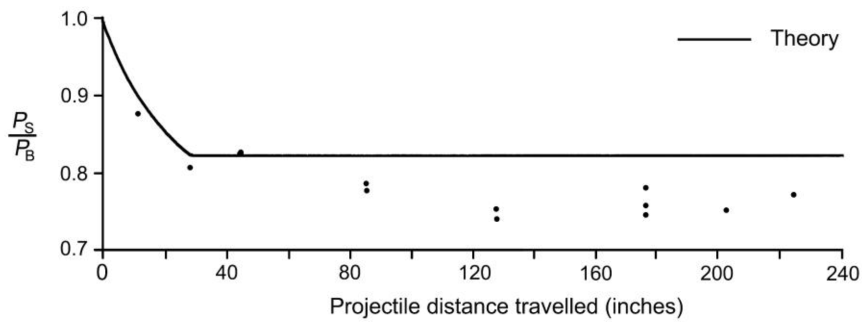

Fig. 8.6 *Projectile base to chamber pressure ratio as a function of distance travelled in a 6 inch BL naval gun for 44 lb propellant charge*

The results for the 44 lb charge are shown in Fig. 8.6. An estimate for Z was made as described above. Parameters are given in Table 8.2.

Table 8.2 *Parameters used in calculating Z for 44lb charge in the 6" BL naval gun*

Charge weight of Cordite SC 205	44 lb
Projectile weight	100 lb
Estimated Propellant vivacity Λ	4.65 /100 bar/sec.
Chamber capacity behind projectile V_C	1500 in^3
Maximum chamber pressure (given)	62900 psi
Muzzle velocity (given)	3370 ft/sec.
Heat loss constant used k	0.17

The ratio of charge weight to projectile weight is now 0.44, so the Lagrange ratio of pressures P_S/P_B is 0.82.

The fit to Eqn. 8.25 is reasonable at the start where the pressure is rising. (Maximum pressure is estimated to occur after the projectile has travelled about 33 inches). The measured projectile base pressures are thereafter rather low compared to theory, but as has been noted this was declared to be a worn barrel and given the exceptionally high pressures for this type of gun, it is expected that gas leakage past the drive-band would have been significant down the length of the barrel.

The same experimental method was used by Heiney in 1979 [11] and by Hansen and Heiney in 1987 [12], but unfortunately there is insufficient detail in these reports to enable an estimate of Z and so a comparison with theory as above to be made.

Jedlicka and Beer [13] used the same experimental method in 2007 on a 30mm anti-aircraft gun, chambered for the Czech 30 x 210mm cartridge. The ammunition used was a training round, type 30mm JNhSv which used the Soviet propellant type Nctp3x1.25/3.5-KF1 This is a single base nitrocellulose propellant in the form of cylindrical kernels with one perforation, and so a 'neutral' burning kernel shape.

Fig. 8.7 shows the experimental results compared to the prediction of Eqn. 8.25. Here, the ratio of charge weight to projectile weight was 0.475. The Lagrange ratio of pressures P_S/P_B is 0.81, which is very similar to that of the 6" naval gun above with a 44 lb charge, but it is notable that in this instance the Lagrange pressure ratio is a good fit to experiment. The measurement near the muzzle is significantly below theory, but this was attributed by the authors to gas leakage due to muzzle erosion in the test barrel.

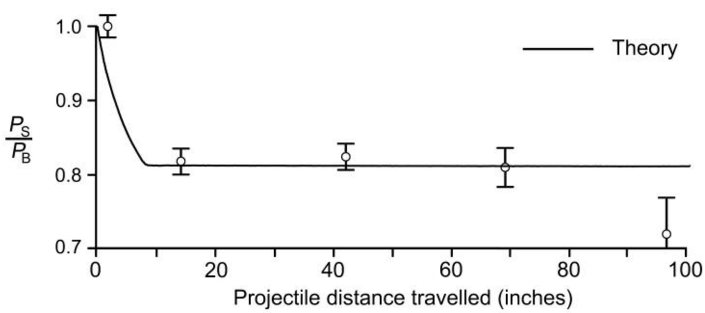

Fig 8.7 *Projectile base to chamber pressure ratio as a function of distance travelled in a 30mm anti-aircraft gun*

An estimation of Z for Eqn. 8.25 was made as described above using the Mayer-Hart system. Parameters used are given in Table 8.3 below

Table 8.3 *Parameters used in calculating Z for the 30mm gun*

Force	3.6×10^6 in-lb/lb
Chamber capacity V_C	14.15 in^3
Ratio of specific heats γ	1.25
Estimated Propellant vivacity Λ	21 /100 bar/sec.
Heat loss constant used k	0.3
Charge weight	0.407 lb
Projectile weight	0.856 lb
Calibre	1.18 in
Projectile travel to muzzle	98 in
Maximum chamber pressure (given)	39,000 psi
Muzzle velocity (given)	3300 ft/sec.

In conclusion it is seen that the pressure gradient is reasonably described by Eqn. 8.25 and the Lagrange solution.

§ 8.8 *The Sébert factor*

Another way of considering the ratio P_S/P_B is attributed to Sébert [14]. Momentum is conserved during the trajectory of the projectile up the barrel between the gun recoiling backwards and the projectile moving forwards,

$$\left[\mathrm{B}+(1-\varepsilon)\mathrm{C}\right]\frac{dx}{dt} = \left[\mathrm{W}+\varepsilon C\right]v_{\mathrm{P}} \qquad 8.31$$

where B be the mass of the gun, ε is the fraction of propellant powder following the projectile up the barrel and x is along the axis of the barrel.

Differentiating with time, $$\left[\mathrm{B}+(1-\varepsilon)\mathrm{C}\right]\frac{d^2x}{dt^2} = \left[\mathrm{W}+\varepsilon C\right]\frac{dv_{\mathrm{P}}}{dt} \qquad 8.32$$

The force accelerating the gun backwards due to the pressure at the back of the chamber is,

$$B\frac{d^2x}{dt^2} = AP_B \qquad 8.33$$

And the force accelerating the projectile up the barrel due to the projectile base pressure is,

$$W\frac{dv_P}{dt} = AP_S \qquad 8.34$$

Now, assume $B+(1-\varepsilon C) \simeq B$ and combine Eqn. 8.32, 8.33 and 8.34.

$$P_B = P_S\left(1 + \varepsilon\frac{C}{W}\right) \qquad 8.35$$

On analysing a large amount of experimental work by Gavre, Sébert determined that the measured muzzle velocities were best fitted to theory if ε should be $^1/_2$. The implication that half the propellant is following the projectile up the barrel at the same velocity as the projectile is specious, but the form of the equation is the same as the Lagrange equation and so it is not unexpected that an experimentally determined value of $^1/_2$ should be obtained for ε.

The Sébert factor was used by French and German ballisticians, but British and American ballisticians always used the Lagrange equation for the pressure gradient. They are, of course, the same, but the assumption that the pressure gradient was due to half the propellent following the projectile up the barrel at the same velocity as the projectile is incorrect and this has led to inappropriate uses of the "Sébert factor".

§ 8.9 *Effective mass*

The concept was developed whereby the effect of a reduced projectile base pressure compared to the chamber pressure could be accounted for by increasing the mass of the projectile in the equation of motion. A fraction ε of the charge weight is added to that of the projectile so that the chamber pressure P_B is acting on an effective mass $W_{eff} = W + \varepsilon C$ to accelerate it up the barrel. The acceleration of the projectile of an effective mass W_{eff} due to the breech pressure P_B is equivalent to the base pressure P_S acting on the projectile alone.

$$\frac{dv}{dt} = \frac{AP_S}{W} \rightarrow \frac{A\,P_B}{W\left(1 + \varepsilon\frac{C}{W}\right)} \rightarrow \frac{A\,P_B}{\left(W + \varepsilon C\right)} \rightarrow \frac{A\,P_B}{W_{eff}} \qquad 8.36$$

If this effective mass is also used in the expression for the kinetic in Résal's energy balance equation, then in § 8.3 it is shown that a value of $\varepsilon = {}^1/_3$ would be a reasonable value when including the mass of the propellant gasses in the effective mass.

However, one way to account for losses due to friction, recoil and heat loss to the barrel in Résal's equation is to assume these losses go as velocity squared and so increase the effective mass in the kinetic energy term. Increasing the value of ε to ${}^1/_2$ from ${}^1/_3$ for the effective mass in Résal's equation actually gives a reasonable approximation for simulating these losses. See Chapter 7 for a fuller discussion of heat loss to the barrel.

When considering the amount of powder following the projectile up the barrel, a notion has gained ground in the popular press that there is a chambrage effect such that $\varepsilon = {}^1/_2$ for 'normal' bottleneck cartridges, $\varepsilon < {}^1/_2$ for magnum cartridges on the basis that comparatively less of the powder will follow the projectile up the barrel, and for straight-walled cartridges $\varepsilon > {}^1/_2$ as more than half the powder will be blown up the barrel behind the projectile. Estimations of the Sébert factor on this basis would seem questionable, as when used in Résal's energy balance equation, it assumes losses would then decrease with the ratio of charge weight to projectile weight, which is not to be expected. Sébert, of course, based his result on the data for a large gun where the chamber diameter was effectively the same as the bore diameter - which is equivalent to a 'straight-walled' cartridge.

§ 8.10 *Adiabatic gas expansion pressure ratio*

An approach proselytised by the American ballistician O.K. Heiney [11] [12] was that the decrease in the pressure at the projectile base was due to the 'quasi isentropic' expansion of the gasses behind the projectile.

$$\frac{P_S}{P_B} = \left[1 - \frac{(\gamma - 1)}{2}\left(\frac{v}{a_0}\right)^2\right]^{\frac{\gamma}{(\gamma - 1)}} \qquad 8.37$$

Where $a_0 = \sqrt{\gamma R T g / M}$ is the speed of sound in the gas at the breech where the gas velocity is zero.

Note that the gas constant R used here (and throughout this book) is in terms of pressure in pounds-weight per unit area and not force per unit area. It is then necessary to introduce the acceleration due to gravity g to ensure the expression is dimensionally correct. The result is that for propellant gasses, $a_0 \approx 43508$ in/sec. or 3625 ft/sec.

This approach is seductive in that the pressure ratio is unity as the projectile starts to move down the barrel, as it should be. For medium performance guns with small ratios of charge to projectile weight, the pressure ratio given by Eqn. 8.37 is happenstantially close to that of Lagrange after all-burnt. See Fig. 8.8.

But Heiney himself [15] finally admitted that this approach significantly undervalues the projectile base pressure in high performance guns such as long range artillery howitzers which have muzzle velocities that exceed 5000 ft/sec. Even a small-arms rifle chambered in 220 Swift, for example, can expect to exceed 4000 ft/sec. using light bullets. Where the velocity in the gun is equal to a_0 then the projectile base pressure would only be just over half the chamber pressure and this is not reconcilable with the observed muzzle velocities in high performance guns.

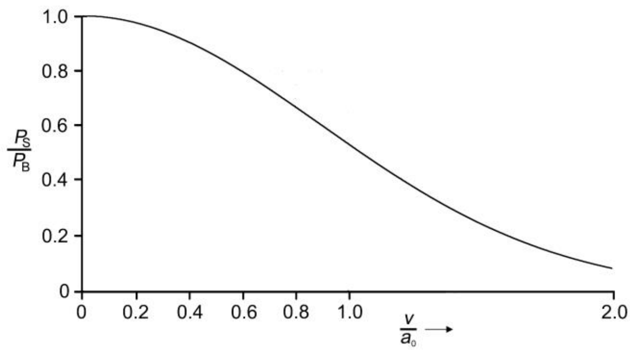

Fig. 8.8 *Pressure against velocity for the adiabatic expansion of a gas in a tube where* $\gamma = 1.25$ *which is typical for nitro propellants.*

Eqn. 8.37 is one of a number of equations which are used to describe the flow conditions for the throat of a Laval nozzle, where the tube diameter is at a minimum and so the rate of change of diameter with direction of gas flow is zero. On this basis, it was assumed that these equations would also apply in a gun, which is a straight-walled tube where the tube diameter is not changing with gas flow. However, the conditions for these equations is that the gas flow is steady-state, where the rate of gas flow is constant with time, and that the expansion is adiabatic. While these condition apply for a rocket nozzle, the experimental evidence is that the pressure ratio in a conventional propellent gun does not obey Eqn. 8.37. It is then not reasonable to say that the conditions of steady-state and adiabatic expansion apply in a conventional propellant gun, certainly before all-burnt.

It would seem, however, that Eqn. 8.37 does apply to air guns. The gas expansion in an air gun can be treated as adiabatic as heat loss to the wall can be neglected. The reservoir volume is usually large compared to the volume of the barrel, and so the reservoir pressure P_B is (to a first approximation) invariant with projectile travel or velocity and so can be treated as a constant in Eqn. 8.37, so satisfying the requirement of being "steady-state". Even in spring air guns, where the mass of gas in the piston is considerably smaller than that in the reservoir of a pre-compressed pneumatic air gun, the inertial mass of the piston is the main determinant of the "breech pressure" P_B and not the increase in volume due to the movement of the projectile down the barrel, as is the case in a conventional propellant gun after all-burnt.

§ 8.11 *The maximum attainable velocity in a conventional propellant gun*

An obvious line of attack is simply to consider the chemical energy in the propellant. Take Résal's equation for the energy balance in the system, where F is the Force of the propellant. The effective mass $W_{eff} = W + C/3$ accounts for the kinetic energy of the propellant gasses as well as that of the projectile. (See § 8.3). If losses are ignored then,

$$PV = FC - (\gamma - 1)\frac{W_{eff}v^2}{2g} \qquad 8.38$$

For a long barrel, the pressure will tend to zero as the gas expands. Assume that to achieve the highest muzzle velocity, the projectile mass is very small compared to the propellant mass. Then in the limit that the pressure reduces to zero, this equation can be rearranged to give the velocity as,

$$v = \sqrt{\frac{6Fg}{(\gamma - 1)}} \qquad 8.39$$

For $\gamma = 1.25$, a typical value of $F = 3835000$ in-lb/lb and $g = 386.4$ in/sec^2, the maximum velocity would be about 190560 in/sec or 15,880 ft/sec. Of course, in reality, heat loss to the barrel, frictional losses of the projectile and the gasses to the barrel walls, and the column of air at one atmosphere pressure in front of the projectile will all take their toll on the practical limits of muzzle velocity.

A series of experiments were performed by Langweiler [16] in the 1930s, in which he fired a rifle loaded with a constant charge weight of fast burning propellant behind a projectile of varying weights. By plotting muzzle velocity vs projectile weight, Langweiler extrapolated a muzzle velocity for zero projectile weight and so estimated a limiting velocity of 9150 ft/sec.

Langweiler proposed a theoretical limit based on Eqn. 8.37 where the limiting velocity is when $P_S = 0$. Using the definition $F = RT/M$ the limiting velocity becomes,

$$v = \sqrt{\frac{2\gamma Fg}{(\gamma - 1)}} \qquad 8.40$$

For the variable values given above, this translates to a limiting velocity of about 121731 inches per second, or 10,144 ft/sec. which is in remarkably good agreement with Langweiler's experimental value.

This is probably happenstance rather than a true alignment of theory and experiment, however. As discussed above, Eqn. 8.37 might be expected to apply where the expansion ratio is close to unity and the chamber pressure P_B remains constant as the projectile travels up the barrel. In Langweiler's case, the fast burning propellant would have been all-burnt before the projectile had travelled any significant distance and the chamber pressure thereafter would have been very dependent on the projectile's travel up the barrel.

It would probably be more correct to propose that Eqn. 8.39 with losses properly attributed would be a better description of the limiting velocity of a conventional propellant gun. For higher velocities, experimenters have turned to light gas guns, where very light projectiles ride the front of a rarefaction wave in gasses with a small molecular weight to maximise the speed of sound in the gas.

Nomenclature

A = bore area: square inches
B = weight of gun: pounds
C = charge weight: pounds
E = energy of propellant gasses: in-lb
F = Force or impetus of the propellant: inch-pounds/inch^3.
g = acceleration due to gravity, which is 386.4 inches/sec^2
M = molecular weight
P = pressure: pounds/inch^2 or psi.
P_B = breech or chamber pressure: pounds/inch^2 or psi.
$\overline{P}$ = mean gas pressure behind the projectile: pounds/inch^2 or psi.
P_S = pressure acting on the base of the projectile: pounds/inch^2 or psi.
t = time: seconds
v = velocity: inches/sec. (Divide by 12 for feet/sec.)
V = volume: inch^3
V_0 = the free space behind the projectile when loaded with propellant: inch^3
V_C = case volume without propellant: inch^3 (252.9 grains of water = one inch^3)
W = mass of the projectile: pounds (7000 grains – one lb)
W_{eff} = the effective mass being accelerated up the barrel: pounds
x = distance travelled up the barrel by the projectile: inches
Z = the fractional amount of charge burnt: dimensionless
β = the burning rate of the powder: inches/sec/psi.
ε = Sébert factor: fraction of charge weight accelerated up the barrel at v_P
γ = the ratio of specific heats for the propellant gasses:
ρ = the density of the propellant gasses: pounds/inch^3
Λ_Z = the vivacity of the propellant: /psi/sec. (often quoted as /100 bar/sec.)

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